A new metric criterion for non-amenability III: Non-amenability of R.Thompson's group F

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ABSTRACT: We present new metric criteria for non-amenability and discuss applications. The main application of the results of this paper is the proof of non-amenability of R.Thompson's group F. This is a continuation of the series of papers on our criteria for non-amenability. The current paper is independent of other papers in the series.

1. Introduction

Amenable groups have been introduced by John von Neumann in 1929 in connection with Banach-Tarski Paradox, although earlier, Banach himself had understood that, for example, the group $\mathbb{Z}$ is amenable. J. von Neumann's original definition states that a discrete group is amenable iff it admits additive invariant probability measure.

In 1950's Folner gave a criterion for amenability, which, for finitely generated groups, can be expressed in terms of the Cayley metric of the group. Using this criterion it is very easy to see that abelian groups are amenable and non-abelian free groups (e.g. $\mathbb{F}_2$) are non-amenable.

In 1980's Grigorchuk [Gr] introduced a new metric criterion for amenability in terms of the co-growth as a refined version of Kesten's criterion [K]. Using this criterion, A.Ol'shanskii [O] constructed a counterexample to von Neumann Conjecture, and using the same criterion S.Adian [Ad1] proved that free Burnside groups of sufficiently large odd exponent are non-amenable. Recently, A.Ol'shanskii and M.Sapir [OS] found a finitely presented counterexample to von Neumann Conjecture where non-amenability of their example relies on non-amenability of free Burnside groups of sufficiently big odd exponent.

In the series of papers including this one, we introduce a different metric criterion (sufficient condition) for non-amenability. Roughly speaking, this criterion is based on verifying that certain type of words in a group do not have small length. This, in a lot of cases, seems manageable to show, compared to measuring the boundary of a finite set. More than that, it was proved(observed) by von Neumann that

if a group contains a copy of $\mathbb{F}_2$ then it is non-amenable. But if the group contains no subgroup isomorphic to $\mathbb{F}_2$ then it is usually hard to prove non-amenability. However, one can continue in the spirit of von Neumann's observaton. For example, from Folner's criterion it easily follows that if a group is quasi-isometric to a group which contains a copy of $\mathbb{F}_2$ then the group is still non-amenable. Our criteria generalizes much further in this direction.

As mentioned above, our criteria relies on verifying that words in a certain class C are not small with respect to Cayley metric in the group [or words in C are not equal identity element; or tend to infinity in length - all these turn out be essentially the same in our case; roughly speaking, words tend to infinity as a result of not being small]. *The class C of words has to be rich enough to enable us to obtain non-amenability results yet subtle enough to allow us to penetrate into the area of studying groups which do not contain $\mathbb{F}_2$ as s subgroup.* Theorems 2.2/2.4. in this paper, Proposition 5, Proposition 6, Proposition 9. in [Ak1], and Theorem 1. in [Ak2] present examples of such classes. As an application of Proposition 9. in [Ak1], we give a new and very short proof of non-amenability of free Burnside groups of sufficiently big odd exponent, and as an application of Theorem 2.4. in this paper we prove non-amenability of R.Thompson's group **F** (Theorem 10.16.) We postpone more general and systematic treatment of the issue to our next publication.

The original definition (and some other popular definitions) of amenability does not involve any metric. However, the (left invariant) metric (or some substitutes) comes up very naturally in questions about amenability.

Definition of Amenability. There exist at least 10 wellknown definitions of the notion of amenable group which look quite different but, amazingly, turn out to be all equivalent. There are also related notions like weak amenability, inner amenability, uniform non-amenability, amenable algebra, amenable action, etc. The equivalences between different definitions of amenable group are often respectable theorems.

The only definition we will be using in this paper is the one obtained from Folner's Criterion. This definition, again, in its own turn has at least 10 different versions which are all equivalent (the equivalences in this case are easy exercises).

Definition 1.1. Let Γ be a finitely generated group. Γ is called amenable if for every $\epsilon > 0$ and finite subsets $K, S \subseteq \Gamma$, there exists a finite subset $F \subseteq \Gamma$ such that $S \subseteq F$ and $\frac{|FK \backslash F|}{|F|} < \epsilon$

The set F is called (ϵ, K)-Folner set. Very often one uses the loose term "Folner set", and very often one assumes K is fixed to be the symmetrized generating set. The set $\{x \in F \mid xK^{-1}K \subseteq F\}$ will be called *the interior of* F, and will be denoted by $Int_K F$. The set $F \backslash Int_K F$ is called *the boundary of* F and will be denoted by $\partial_K F$.

Despite its great theoretical value, Folners' Criterion is often very unpractical. For example, to establish non-amenability, in the direct application of the criterion, one has to argue somehow that given a small $\epsilon > 0$, no set can be an ϵ-Folner set, and it is often very hard to rule out all sets at once since Folner sets may have very tricky, versatile, and complicated geometry and combinatorics. For example, in groups with subexponential growth, one can find a sequence of balls which form a Folner sequence. But in many examples of amenable groups (in generic solvable groups, for example), Folner sets are quite different from the balls. Another major problem is that, given $\epsilon > 0$, the definition does not demand any upper bound on the size of minimal ϵ-Folner set F. Yet another issue is that the size of F is not any explicit function of the size of ∂F.

Acknowledgement: I am grateful to V.Guba and M.Sapir for the very useful discussions related to the content of this paper. In particular, the counterexample pointed out to me by V.Guba (free solvable group on 2 generators of derived length 3), first, convinced me that in the original version the height condition is necessary also in the first theorem (otherwise the claim is false, as the example of V.Guba shows), secondly, understanding the mistake in the proof quickly allowed me not only to correct it but also led to significant simplifications. The major simplifications are the following: a) we do not use quasi-η-normal or weakly quasi-η-normal trees but limit ourselves to the most natural η-normal trees; b) we use height function for dividing Folner sets into "annuli" which simplifies many things in the arrangement of ξ-partners; in particular, naturally, we count the height not from the origin of the each tree separately but but from the origin of the group for all trees. As a consequence, in the condition (D), we allow also even exponents, not just odd. In the original version, it was this stronger (but simpler looking) condition that we were indeed proving in the second part of the paper. In the original version, our claim that the tree with the

root in the annulus S_i does not intersect S_{i+1} was not correct, more precisely, the arrangment we had presented does not guarantee that. Using height function for dividing Folner sets into annuli simplifies the picture greatly, and many technical issues one had to deal with in earlier version simply disappear. We would like to remark that condition (C) separates R.Thompson's group $\mathbf{F}$ from $\mathbb{Z} \wr \mathbb{Z}$, its close amenable friend, and the height codnition (D) very strongly separates $\mathbf{F}$ further from its other amenable relatives, namely, free solvable groups of derived length $d \geq 2$. We refer the reader to the last section for the relevant discussion.

Structure of the paper: The paper consists of two parts. In Part 1, we prove general criteria for non-amenability. In Part 2, we discuss the application of these results to R.Thompson's group $\mathbf{F}$. The two parts are independent of each other except in Part 2, we verify that $\mathbf{F}$ satisfies all conditions of the theorem in Part 1. The notations of the two parts are also totally unrelated. We have attached 9 Figures at the end of the paper. The paper is fairly self-contained. We use some basic results from [CFP].

Part One: Non-Amenability Criterion.

2. General non-amenability theorems.

In this section, we will state our main theorems which the non-amenability criteria we will be discussing in this paper. First, we introduce the notion of height function:

Definition 2.1 (Height function)**.** Let G be a group. A function $h : G \to \mathbb{N} \cup \{0\}$ is called a height function on G, if $h(xy) \leq h(x) + h(y)$ for all $x, y \in G$; moreover, $h(x) > 0$ unless x is the identity element.

A good example of a height function $h(x)$ is the function $|x|$ representing the length of the element in the Cayley metric.

Theorem 2.2. *Let Γ be a finitely generated group, $\xi, \eta \in \Gamma$, $\pi : \Gamma \to \Gamma/[\Gamma, \Gamma]$ be the ablieanization epimorphism. Let also H be a cyclic subgroup generated by η, and H' be a cyclic subgroup generated by ξ. Let also $h : \Gamma \to \mathbb{N} \cup \{0\}$ be a height function, $h(\xi) = h(\xi^{-1}) = h(\eta) = h(\eta^{-1}) = 1$. Assume that the following conditions are satisfied:*

$(A)\pi(\xi), \pi(\eta)$ generate a subgroup isomorphic to $\mathbb{Z}^2$.

$(B)\forall k \in \mathbb{N}, \epsilon_j \in \mathbb{Z}_{odd}, p_i \in \mathbb{Z}\backslash\{0\}, 1 \leq j \leq k, 1 \leq i \leq k-1, p_0, p_k \in \mathbb{Z}$ we have

$$\eta^{p_0}\xi^{\epsilon_1}\eta^{p_1}\xi^{\epsilon_2}\dots\eta^{p_{k-1}}\xi^{\epsilon_k}\eta^{p_k} \notin H \cup H'$$

where at most one of the numbers in the sequence $(p_0, p_1, \dots, p_k)$ is even.

(D) there exists an odd integer $400 < p < 410$ such that for all $g \in \Gamma$,

(i) for all but possibly one $\delta \in [-100, 100]$, the inequality $h(g\eta^{\delta}\xi^{-p}) > h(g) + 100$ is satisfied.

(ii) if for some $\delta_0 \in (-100, 100)$ the inequality of (i) is not satisfied (i.e. if we have $h(g\eta^{\delta_0}\xi^{-p}) \leq h(g) + 100$), then for all $\delta \in (-100, \delta_0]$, the following inequality is satisfied: $h(g\eta^{\delta}\xi^{p}) > h(g) + 100$.

Then Γ is not amenable.

Remark 2.3. By passing to a subgroup if necessary, we may and will assume that ξ and η generate Γ

The proof of Theorem 2.2. will be easily modified to prove the following generalization of it:

Theorem 2.4. *Let Γ be a finitely generated group, $\xi, \eta \in \Gamma$, $\pi : \Gamma \to \Gamma/[\Gamma, \Gamma]$ be the ablieanization epimorphism. Let also $h : \Gamma \to \mathbb{N} \cup \{0\}$ be a height function, $h(\xi) = h(\xi^{-1}) = h(\eta) = h(\eta^{-1}) = 1$. Assume that the conditions $(A), (D)$ and the following condition are satisfied:*

(C) for all $n \in \mathbb{N}$ and for all $\epsilon_i, \delta_i \in \mathbb{Z}_{odd}, 1 \leq i \leq n$, we have $\eta^{\delta_1}\xi^{\epsilon_1}\dots\eta^{\delta_n}\xi^{\epsilon_n} \neq 1$.

Then Γ is not amenable.

Theorem 2.4. applies to R.Thompson's group **F** to establish it's non-amenability. We discuss this application in later sections.

Remark 2.5. We would like to make clear that, in condition (D), if the inequality $h(g\eta^{\delta}\xi^{-p}) > h(g)+100$ is satisfied for all $\delta \in (-100, 100)$ (but may fail for $\delta = -100$ or for $\delta = 100$) then we do not demand the condition of part (ii).

Remark 2.6. Notice that conditions (A)-(B) imply condition (C) as well, so (A) and (C) are common in both theorems.

Condition (C) is extremely fundamental in our approach but of course not sufficient by itself alone (or with (A)) to establish non-amenability as we will see in later sections. *It would be easier for the reader to focus first on Theorem 2.2. since certain constructions in the proof could be more intuitive and simpler in this case.*

Remark 2.7. Notice that condition (A) alone implies that $\xi^n \neq \eta^m$ for all integers m, n unless $m = n = 0$. This is kind of little version of (B) or (C). Notice also that conditions (A) and (C) imply the following condition which resembles condition (B):

$(E) : \forall k \in \mathbb{N}, \epsilon_j \in \mathbb{Z}_{\mathrm{odd}}, p_i \in \mathbb{Z}, 1 \leq j \leq k, 0 \leq i \leq k$ we have

$$\eta^{p_0}\xi^{\epsilon_1}\eta^{p_1}\xi^{\epsilon_2}\ldots\eta^{p_{k-1}}\xi^{\epsilon_k}\eta^{p_k} \notin H \cup H'$$

where none of the numbers in the sequence $(p_0, p_1, \ldots, p_k)$ is even.

One can also consider the following height condition:

(D'): for all $n \in \mathbb{N},\ \epsilon_i \in \mathbb{Z}\backslash\{0\},\ \delta_i \in [-100, 100]\backslash\{0\},\ 1 \leq i \leq n$, $g = \eta^{\delta_1}\xi^{\epsilon_1}\ldots\eta^{\delta_n}\xi^{\epsilon_n} \in \Gamma$, and for all $\delta \in [-100, 100]\backslash\{0\}, \epsilon \in \mathbb{Z}\backslash\{0\}$ the inequality $h(g\eta^{\delta}\xi^{\epsilon}) > h(g)$ is satisfied.

One can prove that $(A) + (C) + (D') \Rightarrow$*Non-amenability.*

This result is very appealing because (D') looks much simpler than (D). However, this is not a new result since (D') implies existence of non-abelian free subgroup. The idea is that, one can modify the condition (D') to obtain new conditions which would be still sufficient to yield non-amenability results. Condition (D) of Theorem 2.4. is just one example of such a modification.

It is also useful to think of the comparison between (D) and (D') as *dynamical condition* vs. *rigid condition.* We have taken this approach in our earlier works as well. For example, in [Ak1], conditions $(P), (TO)$ are rigid, and conditions $(P'), T_{p,q}, TO_{p,q}$ are dynamical. It is common in our series of papers that rigid conditions imply the existence of free subgroups but using dynamical conditions one can obtain non-amenability results for group without the subgroup isomorphic to $\mathbb{F}_2$.

Remark 2.8. The fact that we use arbitrary height function makes the claims of the theorems not only very strong but also provides great flexibility in applications. For example, very often very little is known about the Cayley metric of the group, so one can work with the most convenient height function instead.

3. Generalized binary trees.

We will need some notions about *binary trees* and *generalized binary trees* of groups. Let F be a finite subset of a finitely generated group Γ. Let us remind that binary tree is a tree such that all vertices have valence 3 or 1 and one of the vertices of valence 3 is marked and called a root.

Definition 3.1 (Binary Trees)**.** A binary tree $T = (V, E)$ of F is a finite binary tree such that $V \subseteq F$. A root vertex of T will be denoted by $r(T)$. Vertices of valence 3 are called internal vertices and vertices of valence 1 are called end vertices. The sets of internal and end vertices of T are denoted by $Int(T)$ and $End(T)$ respectively.

Definition 3.2. (see Figure 1) A generalized binary tree $T = (V, E)$ of F is a finite tree satisfying the following conditions:

(i) All vertices of T have valence 3 or 1. Vertices of valence 3 are called *internal vertices*, and vertices of valence 1 are called *end vertices*.

(ii) All vertices of T either consist of triples (i.e. subsets of cardinality 3) or single elements of F. If a vertex has valence 3 then it is a triple; if it has valence 1 then it is a singleton. For two distinct vertices $u, v \in V$, their subsets, denoted by $S(u), S(v)$, are disjoint. The union of all subsets (triples or singletons) representing all vertices of T will be denoted by $S(T)$

(iii) One of the vertices of T is marked and called the root of T. The root always consists of a triple and has valence 3, and it is always an internal vertex. We denote the root by $r(T)$.

(iv) For any finite ray $(a_0 = r(T), a_1, a_2, \ldots, a_k)$ of T which starts at the root, and for any $i \in \{0, 1, 2, \ldots, k\}$ a vertex a_i is called a *a vertex of level i*.

(v) One of the elements of each triple vertex is chosen and called *a central element*, the other two elements are called *side elements*.

(vi) If a is a vertex of level $n \in \mathbb{N}$ of T of valence 3, and b, c are two adjacent vertices of level $n + 1$, then the set of all vertices which are closer to b than to a form a *branch of T beyond vertex b*. Similarly we define a branch beyond vertex c.

(vii) Similar to (vi), we define the branches beyond the root vertex. So the tree consists of the root and the three branches beyond the root.

Definition 3.3. A generalized binary tree T of $F \subset \Gamma$ is called trivial if it has only 4 vertices, i.e. one root vertex and 3 end vertices.

Definition 3.4. If $T = (V, E)$ is a generalized binary tree of $F \subset \Gamma$, $A \subseteq V$, then the union of all subsets (triples or singletons) which represent the vertices of A will be denoted by $S(A)$. In particular, the union of all subsets representing all vertices of T will be denoted by $S(T)$.

Remark 3.5. Notice that $S(A) \subseteq F$ for all $A \subseteq V$.

Definition 3.6. The set of end vertices of a generalized binary tree $T = (V, E)$ will be denoted by $End(T)$, and the set of internal vertices will be denoted by $Int(T)$. Also, $Single(T), Central(T), Side(T)$ denote the set of all singleton vertices, central elements and side elements respectively.

Remark 3.7. By the definition of a generalized binary tree, $Single(T) = End(T)$.

Definition 3.8. For a generalized binary tree $T = (V, E)$, for all $v \in V \backslash \{root(T)\}$, $p(v)$ denotes the vertex which is adjacent to v such that $level(p(v)) = level(v) - 1$; and for all $v \in V \backslash End(T)$, $n(v)$ denotes the set of vertices v' which are adjacent to v such that $level(v') = level(v) + 1$.

Remark 3.9. n stands for *next*, p stands for *previous*.

We will need the following

Lemma 3.10. *If* $T = (V, E)$ *is a GBT then*

$$|S(End(T))| \geq \frac{1}{4}|S(T)|$$

Proof. For the trivial generalized binary tree we have $|S(T)| = 6, |S(End(T)| = 3$ so the inequality is satisfied. Let T be any non-trivial GBT, and $v \in End(T)$, $level(v)$ is maximal, $w = p(v)$. By definition, v is a singleton and w is a triple vertex.

Let $y \in p(w)$. By definition, y is a singleton and two vertices in $n(y)$ are pair vertices. We can delete all vertices and edges in the branch beyond y to obtain a new GBT T'.

Then $|S(T')| = |S(T)| - 4, |S(End(T'))| = |S(End(T))| - 1$, since $1/4 \geq 1/4$ we may proceed by induction on $|S(T)|$. $\square$

Colored Generalized Binary Trees. For the group $\mathbb{Z}^2$, we call an element $(a, b) \in \mathbb{Z}^2$ white if a and b have different parity; if a and b have the same parity then we call (a, b) black (so this is like a coloring of the chessboard).

Definition 3.11 (Colored elements)**.** Let Γ be a finitely generated group satisfying condition (A). An element $\gamma \in \Gamma$ is called *white* if $\pi(\gamma)$ is white; it is called *black* if $\pi(\gamma)$ is black.

Definition 3.12 (Colored trees)**.** Let T be a GBT of $F \subset \Gamma$. T is called black (white) if the central elements of all triple vertices are black (white), and the side elements of all triple vertices are white (black). T is called normal if it is either black or white. A ray (a branch) of a normal tree is called black (white) if the central elements of all triples of the ray (branch) are black (white). (see Figure 2.)

Definition 3.13 (Normal, η-normal trees)**.** Let Γ be a finitely generated group satisfying condition (A). A GBT T of $F \subset \Gamma$ is called *normal* if it is either black or white. T is called *η-normal* if it is normal and all triple vertices v are of the type: $\{x_v, x_v h_1, x_v h_2\}$ where $x_v \in F$ and $h_1 = \eta^{n_1}, h_2 = \eta^{n_2}, n_1, n_2 \in \mathbb{Z}_{\text{odd}}$.

Labeled Generalized Binary Trees: We will introduce a bit more structure on generalized binary trees. This structure is needed to explain that in tree building process, in the proof of Theorem 2.2, we do satisfy condition (ii) in Definition 3.2, i.e. we do not get any loops.

Let Γ be a finitely generated group satisfying condition (A), $F \subset \Gamma$ be a finite subset of Γ, $F_0 \subseteq F$ is partitioned into 2-element subsets $\{x, y\}$ such that $y \in \{x\xi, x\xi^{-1}\}$. Thus every element in F_0 has a ξ-partner which we denote by $N_\xi(x)$ and since ξ is fixed, we will drop it and denote by $N(x)$. By definition, $N(N(x)) = x, \forall x \in F_0$.

Let $T = (V, E)$ be a generalized binary tree of F, $r = (b_1, b_2, \ldots, b_k)$ be a finite ray in T, such that $level(b_i) = level(b_{i-1}) + 1)$, $\forall i \in \{2, 3, \ldots, k\}$. We will associate $W(r) \in \Gamma$ to r.

Assume that $e \in E(T)$ is an edge connecting $v, w \in V(T)$ such that $level(w) = level(v) + 1$. Assume also that the following conditions are satisfied:

(L1) there exists $a \in S(v), b \in S(w)$ such that $a = N(b)$.

(L2) the element b is the central element of w, and the elements of $S(w) \backslash \{b\}$ are side elements.

Then we label the edge e by the element $a^{-1}b$. Thus all edges $e_1, e_2, \ldots, e_{k-1}$ are labeled by some elements $l(e_1), l(e_2), \ldots, l(e_{k-1})$.

On the other hand, each vertex $b_i, 1 \leq i \leq k-1$ is a triple vertex, and let $b_i = (x_i, y_i, z_i)$ with x_i being a central element. Also, if b_k is a triple vertex then let x_k be its central element, and if b_k is a singleton vertex then let x_k be its only element.

Then either $y_i = N(x_{i+1})$ or $z_i = N(x_{i+1})$, for every $1 \leq i \leq k-1$. In the first case, we label the vertex b_i by $x_i^{-1}y_i$ and in the second case, we label the vertex b_i by $x_i^{-1}z_i$.

Thus all vertices $b_1, b_2, \ldots, b_{k-1}$ are labeled by some elements $l(b_1), l(b_2), \ldots$ $l(b_{k-1})$.

Then we label the ray r by the element $l(b_1)l(e_1)l(b_2)l(e_2) \ldots l(b_{k-1})l(e_{k-1})$, and we will denote this element by $L(r)$.

Definition 3.14. The generalized binary tree $T = (V, E)$ of $F \subset \Gamma$ is called *labeled* if every edge $e = (v, w) \in E$ with $p(w) = v$ satisfies conditions (L1) and (L2) and every edge, vertex, and level increasing ray are assigned elements of Γ as described above.

Remark 3.15. Notice that the labeling structure of a generalized binary tree of F depends on the choice of non-torsion element $\xi \in \Gamma$ and subset $F_0 \subset F$ which can be partitioned into pairs of ξ-partners, and it depends on the partitioning as well.

Lemma 3.16. *Let Γ satisfies conditions (A) and (B), $F \subset \Gamma$ be a finite subset, $F_0 \subseteq F$ be partitioned into pairs of ξ-partners. Let also $T = (V, E)$ be a η-normal labeled generalized binary tree (with respect to ξ, F_0, and partitioning).*

Let r_1 and r_2 are two different level increasing rays of T with the same starting vertex w and different ending vertices w_1 and w_2. Then for every $x \in S(w_1)$ and for every $y \in S(w_2)$ the condition $x^{-1}y \notin H$ holds.

Proof. Let $r_1 = (b_1, b_2, \dots, b_k), r_2 = (c_1, c_2, \dots, c_l)$ where $b_i, 1 \leq i \leq k, c_j, 1 \leq j \leq l$ are vertices of the rays. Denote $e_i = (b_i, b_{i+1}), 1 \leq i \leq k-1,\ e'_j = (c_j, c_{j+1}), 1 \leq j \leq l-1$. By assumption, $b_1 = c_1 = w$, and we may assume that $b_2 \neq c_2$.

Notice that, because the tree T is η-normal, we have $l(b_i) = \eta^{2p_i+1}, p_i \in \mathbb{Z}$ and $l(c_j) = \eta^{2q_j+1}, q_j \in \mathbb{Z}$ for all $1 \leq i \leq k-1, 1 \leq j \leq l-1$. Also, notice that $l(e_i), l(e'_j) \in \{\xi, \xi^{-1}\}$ for all $1 \leq i \leq k-1, 1 \leq j \leq l-1$.

On the other hand, let $w = (\alpha, \beta, \gamma)$, and without loss of generality, assume $\beta = N(b), \gamma = N(c)$ where b and c are central elements of vertices b_2 and c_2 respectively.

Since the tree T is η-normal, we have $\beta^{-1}\gamma = \eta^n$ where n is some non-zero integer (not necessarily odd, since we do not know which of the elements α, β, γ is central!)). Then $x^{-1}y = L(r_1)^{-1}\eta^n L(r_2)$, but by the condition (B), we have $L(r_1)^{-1}\eta^n L(r_2) \notin H$. $\square$

4. Zigzags.

In this section we will assume that Γ is a finitely generated group satisfying condition (A). A good and useful example of such a group is the group $\mathbb{Z}^2$ itself.

Definition 4.1. Let $H_{odd} = \{\eta^{2n+1} \mid n \in \mathbb{Z}\},\ H'_{odd} = \{\xi^{2n+1} \mid n \in \mathbb{Z}\}$. For every $x \in \Gamma$, we call the sets xH and xH' *the horizontal lines* and *the vertical lines*, respectively, *passing through* x. A sequence $(x_1, x_2, \dots, x_m)$ of elements of Γ will be called *a zigzag* if for all $1 \leq i \leq m-2$, either $x_i^{-1}x_{i+1} \in H_{odd}$ or $x_i^{-1}x_{i+1} \in H'_{odd}$; moreover, $x_i^{-1}x_{i+1} \in H_{odd} \Rightarrow x_{i+1}^{-1}x_{i+2} \in H'_{odd}$ and $x_i^{-1}x_{i+1} \in H'_{odd} \Rightarrow x_{i+1}^{-1}x_{i+2} \in H_{odd}$. The number m will be called *the length of* Z.

Notice that given any horizontal line L and any zigzag $Z = (x_1, x_2, \dots, x_m)$ in Γ, if $m \geq 3$ and $x_1 \notin L$ then we have $|Z \cap L| = 2n, n \in \mathbb{N}$. Assume $|Z \cap L| \geq 2$, and let $i(Z, L) = min\{j \mid x_j \in L\}$.

Definition 4.2. A zigzag Z is called black (white) if $x_{i(Z,L)}$ is black (white) for some (consequently, for any) horizontal line L.

Remark 4.3. By condition (A), any zigzag of cardinality at least 3 is either black or white, and the notion does not depend on the choice of L.

Remark 4.4. If the condition (C) is also satisfied, then $|Z \cap L| \leq 2$ for any zigzag Z and any horizontal (vertical) line L. Indeed, if the zigzag Z leaves L at a black (white) element, then, by condition (A), it may come back to hit L again at a white (black) element only, but this is impossible by condition (C).

Remark 4.5. Notice that if zigzag $(\ldots, x_1, \ldots, x_n, \ldots)$ is black(white) then the zigzag $(\ldots, x_n, \ldots, x_1, \ldots)$ is white(black). So choosing the color is equivalent to choosing orientation.

Lemma 4.6. *Let $Z_1 = (x_1, x_2, \ldots), Z_2 = (y_1, y_2, \ldots)$ be two black zigzags in Γ, Γ satisfies condition (B), x_1, y_1 are white elements belonging to the same horizontal, $x_1 \neq y_1$. Then $Z_1 \cap Z_2 = \emptyset$(see Figure 5.)*

Of course, the claims of the lemma immediately follow from condition (B). Lemma 4.6. has already been used, essentially, in the proof of Lemma 3.16 (see the last paragraph of the proof). We mention this lemma because it is useful to think about the proof of Lemma 3.16. in new terms.

Generalized Segments. We will be working with tilings of the group Γ into segments or generalized segments.

Definition 4.7 (Segments)**.** Let $x, y \in \Gamma$ belong to the same horizontal or vertical line L. Then by $seg(x, y)$ we denote the finite set of all points (elements) in between x and y including x and y, and by $int(x, y)$ we denote the set of points in between x and y excluding x and y. A finite subset I of L is called a segment (an interval) if there exists $x, y \in I$ such that $I = seg(x, y)$ $(I = int(x, y))$.

Definition 4.8 (Generalized Balanced Segments)**.** A finite subset I will be called *a generalized balanced segment* if
(i) $I \subset L$ for some horizontal line L.
(ii) $|I|$ is divisible by 12.
(iii) there exist $x, y \in I$ such that $y = x\eta^n, n \leq 36$ and $I \subseteq seg(x, y)$
(iv) I contains same number of black and white elements.

Definition 4.9 (Rightmost element)**.** Let I be a generalized segment. Then there exists a unique element $z_0 \in I$ such that for any $z \in I$ there exists a non-positive integer n such that $z = z_0\eta^n$. The element z_0 will be called *the rightmost element* of I.

Foliations of groups by zigzags. We will be interested in *foliating* groups with zigzags, although we would like to remind that a zigzag Z is a very discrete object. By foliation we simply mean partitioning, although it is useful to think of this partitioning as foliation because of the associations it creates.

Definition 4.10. A partition $\Gamma = \sqcup_{i \in I} Z_i$ into two-sidedly infinite zigzags $Z_i, i \in I$ is called *a foliation*.

Zigzags of the foliation will be also called *leafs* of it, and we will be using both of the terms *zigzag* and *leaf* interchangeably.

Definition 4.11 (Compatibility of the foliation with tiling)**.** Let Fol_0 be a foliation of Γ, and $\{I(x)\}_{x \in X}$ be a tiling of Γ by generalized balanced segments. We say a zigzag Z is compatible with the tiling $\{I(x)\}_{x \in X}$ if each horizontal piece of Z is a subset of $I(x)$ for some $x \in X$. A foliation Fol_0 is called compatible with the tiling $\{I(x)\}_{x \in X}$ if each leaf of Fol_0 is compatible.

Definition 4.12 (Compatibility of GBT with tiling)**.** Let $\{I(x)\}_{x \in X}$ be a tiling of Γ by generalized balanced segments. A GBT T of Γ is called compatible with the tiling, if for all $v \in V(T)$, there exists $x \in X$ such that $S(v) \subset I(x)$.

Remark 4.13. If Γ satisfies condition (C), and Fol_0 is a foliation of Γ compatible with the tiling $\{I(x)\}_{x \in X}$ then each leaf Z of the foliation intersects each segment $I(x)$ at most once, i.e. $|Z \cap I(x)| \leq 2$.

We would like to conclude this section with introducing some notions for labeled η-normal generalized binary trees.

Definition 4.14 (starting element, tail zigzag)**.** Let T be a labeled η-normal black (white) GBT, where the triple (a, b, c) is a root, and a is the central element. The element $N(a)$ will be called *the starting element* of T and denoted by $start(T)$. Notice that there is a unique maximal (in length) sequence $(x_1, x_2, x_3, \ldots, x_n)$ in $S(T)$ such that the following conditions are satisfied:
(i) $x_1 = x, x_2 = N(x)$.
(ii) x_i is black if and only if i is odd.
(iii) $x_{2i} = N(x_{2i-1})$ for all $1 \leq i \leq [n/2]$.
We will call this maximal sequence *the tail zigzag* of T.

In the above definition, it is also clear that the number n is necessarily even, since the zigzag consists of pairs of ξ-partners. Let $r = (v_0, v_1, v_2, \ldots, v_{[n/2]})$ be the ray such that $v_0 = root(T), x_{2i} \in v_i, 1 \leq i \leq [n/2] - 1$, and $x_{[n/2]} \in v_n$. We will call the ray r the tail of T and will denote it by $tail(T)$.

Definition 4.15 (special GBTs)**.** A labeled black (white) η-normal GBT is called special if $\{a'\}$ is an end vertex where a' is the starting element of T (see Figure 6).

In other words, T is called special if $\{N(a)\}$ is an end vertex where a is the central element of the root. The tail of a special tree T has minimal length 2.

5. Combinatorial Lemmas.

In this section we will prove combinatorial lemmas which may look very technical but are very simple in essence. Let I be the interval in $\mathbb{Z}$ of length $12n, n \in \mathbb{N}$. Subsets of I of cardinality three are called *triples.* If one element of the triple is even (odd) and the other two are odd (even) then the triple is called *black (white).*

Assume we want to cover the segment I with disjoint colored triples. If we use only triples of one color, then we cover at most three quarters of I. To cover a higher percentage, we need to use triples of both colors. It is curious to observe that using only one black triple and then only white triples does not help either, so we need at least two triples of each color. *Notice also that if we start covering the segment I with disjoint triples (some white, some black) in arbitrary fashion, to the point so that we cannot add a new triple (i.e. we do not have one black and two white uncovered elements, and we do not have one white*

and two black uncovered elements) then we still would have covered at least three quarters of I.

Following the definition of balanced segments in Section 4, we would like to introduce the notion of balanced subset of $\mathbb{Z}$ as follows:

Definition 5.1. A finite subset $I \subset \mathbb{Z}$ is called balanced if $|I|$ is divisible by 12, and I contains the same number of even and odd elements.

We would like to start with a simple but useful

Lemma 5.2. *Let $I \subset \mathbb{Z}$ be a balanced segment, and $\Delta_1, \dots, \Delta_k \subset I$ be mutually disjoint white triples in I. If there exists a black triple $\Delta_b \subseteq I \backslash \sqcup_{i=1}^k \Delta_i$ then there also exists a white triple $\Delta_w \subseteq I \backslash \sqcup_{i=1}^k \Delta_i$.*

Proof. Let I_b, I_w denote the set of black and white elements of I respectively. Notice that $|(\sqcup_{i=1}^k \Delta_i) \cap I_w| = k, |(\sqcup_{i=1}^k \Delta_i) \cap I_b| = 2k$.

Assume that there exists a black triple $\Delta_b \subseteq I \backslash \sqcup_{i=1}^k \Delta_i$. Then $2k \leq \frac{1}{2}|I| - 1$. Since $4 \mid |I|$ we obtain $2k \leq \frac{1}{2}|I| - 2$. Then $|I_b \backslash \sqcup_{i=1}^k \Delta_i| \geq 2$. Since $|I_w \backslash \sqcup_{i=1}^k \Delta_i| \geq \frac{1}{2}|I| - k \geq \frac{1}{4}|I| + 1$ we have a white triple as well in $\Delta_w \subseteq I \backslash \sqcup_{i=1}^k \Delta_i$. $\square$

For the next two lemmas, let, for all $1 \leq r \leq k$, $A_r \subset \mathbb{Z}$ be balanced subsets, $A_{i_1} \cap A_{i_2} = \emptyset, \forall i_1, i_2 \in \{1, 2, \dots, k\}, i_1 \neq i_2$, $A = \sqcup_{1 \leq i \leq k} A_i$.

We also denote $O(A) = \mathbb{Z}_{\text{odd}} \cap A, E(A) = \mathbb{Z}_{even} \cap A$.

The next lemma is not used directly in the proofs, but we hope that, despite its technicality, it will provide a useful intuitive background. The reader may wish to skip the (statement and) proof of it. For this lemma we also assume that $N = min_{1 \leq i \leq k}|A_i| \geq 10000$.

Lemma 5.3. *Let $B_1, B_2, \dots, B_p, C_1, C_2, \dots, C_q$ be mutually disjoint subsets of $A = \sqcup_{1 \leq r \leq k} A_i$ of cardinality 3, such that each of these triples is a subset of A_i for some $1 \leq i \leq k$ (i depending on the set).*

Let also B be the set of all central elements of $B_i, 1 \leq i \leq p$, and let $S = \sum_{1 \leq r \leq k} min\{|B \cap A_r|, \frac{1}{3}|A_r| - |B \cap A_r|\}$.

Also, assume that for each $1 \leq r \leq k$, either $|(O(A) \cap A_r) \backslash (\sqcup_{1 \leq i \leq p} B_i) \sqcup (\sqcup_{1 \leq j \leq q} C_j)| \leq 2$ or $|(E(A) \cap A_r) \backslash (\sqcup_{1 \leq i \leq p} B_i) \sqcup (\sqcup_{1 \leq j \leq q} C_j)| \leq 2$. And finally, assume that $S \geq \frac{1}{10}|A|$.

Then $|(\sqcup_{1 \leq i \leq p} B_i) \sqcup (\sqcup_{1 \leq j \leq q} C_j)| > (\frac{3}{4} + \frac{1}{100})|A|$.

Proof. Let $\delta_r = \frac{min\{|B\cap A_r|, \frac{1}{3}|A_r| - |B\cap A_r|\}}{|A_r|}, 1 \leq r \leq k$. Then $0 \leq \delta_r \leq 1/3$.

Just to exercise on the notations, let us discuss the case $\delta_r = 0$ for some $r \in \{1, \ldots, k\}$. For simplicity let us also assume that either $(O(A) \cap A_r) \subseteq (\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)$ or $(E(A) \cap A_r) \subseteq (\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)$.

If $\delta_r = 0$ then either $|B \cap A_r| = 0$ or $|B \cap A_r| = \frac{1}{3}|A_r|$.

Assume $|B\cap A_r| = 0$. Then $(\sqcup_{1\leq i\leq p} B_i) \cap A_r = \emptyset$. Then $|((\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)) \cap A_r| = |(\sqcup_{1\leq j\leq q} C_j) \cap A_r| \leq \frac{3}{4}|A_r|$.

But since either $(O(A) \cap A_r) \subseteq (\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)$ or $(E(A) \cap A_r) \subseteq (\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)$, and moreover, $(\sqcup_{1\leq i\leq p} B_i) \cap A_r = \emptyset$, we have $|(\sqcup_{1\leq j\leq q} C_j) \cap A_r| = \frac{3}{4}|A_r|$. Then $|((\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)) \cap A_r| = \frac{3}{4}|A_r|$.

Now assume $|B \cap A_r| = \frac{1}{6}|A_r|$. Then $(\sqcup_{1\leq j\leq q} C_j) \cap A_r = \emptyset$. Then, similarly, we obtain $|((\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)) \cap A_r| = \frac{3}{4}|A_r|$.

Thus in both cases we obtained that $|((\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)) \cap A_r| = \frac{3}{4}|A_r|$. The simple idea for the proof of the lemma is that if $\delta_r > 0$ then we can have A_r covered with triples of both types B_i or C_j so that the percentage of both types is bigger than 0 (moreover, isolated from 0), which will enable us to cover more than three quarters of A_r. More precisely, we will prove that $|((\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)) \cap A_r| = (\frac{3}{4} + c\delta_r - \frac{1}{100})|A_r|$ where c is a uniform positive constant sufficiently far away from 0.

Now we are considering the general case $\delta_r \geq 0$.

Case 1. $|B \cap A_r| \leq \frac{1}{3}|A_r| - |B \cap A_r|$.

Then $|B \cap A_r| \leq \frac{1}{6}|A_r|$. Then $|(\sqcup_{1\leq i\leq p} B_i) \cap A_r| \leq \frac{1}{2}|A_r|$. Then the condition $|(O(A) \cap A_r)\backslash(\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)| \leq 2$ does not hold, and we have $|(E(A) \cap A_r)\backslash(\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)| \leq 2$. [If $|B\cap A_r| \geq \frac{1}{3}|A_r| - |B\cap A_r|$ then we would have $|(O(A)\cap A_r)\backslash(\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)| \leq 2$].

Then $|(\sqcup_{1\leq i\leq p} B_i) \cap A_r| = 3\delta_r|A_r|$ and $|(\sqcup_{1\leq j\leq q} C_j) \cap A_r| \geq 2(\frac{1}{2}(\frac{1}{2} - \delta_r)|A_r| - 3) = \frac{3}{2}(\frac{1}{2} - \delta_r)|A_r| - 9 \geq \frac{3}{2}(\frac{1}{2} - \delta_r)|A_r| - \frac{1}{1000}|A_r| = (\frac{3}{4} - \frac{3}{2}\delta_r - \frac{1}{1000})|A_r|$.

Then $|((\sqcup_{1\leq i\leq p} B_i) \sqcup (\sqcup_{1\leq j\leq q} C_j)) \cap A_r| \geq (3\delta_r + \frac{3}{4} - \frac{3}{2}\delta_r - \frac{1}{1000})|A_r| = (\frac{3}{4} + \frac{3}{2}\delta_r - \frac{1}{1000})|A_r|$.

Case 2. $|B\cap A_r| \geq \frac{1}{3}|A_r| - |B\cap A_r|$.

Then we have $|B\cap A_r| \geq \frac{1}{6}|A_r|$ and $|(O(A)\cap A_r)\backslash(\sqcup_{1\leq i\leq p}B_i)\sqcup(\sqcup_{1\leq j\leq q}C_j)| \leq 2$.

Let C denotes the set of central elements of $C_j, 1\leq j\leq q$. Then $|B\cap A_r| = (\frac{1}{3}-\delta_r)|A_r|$ and $|C\cap A_r| = \delta_r|A_r|$. Similarly, in this case we obtain that

$$|((\sqcup_{1\leq i\leq p}B_i)\sqcup(\sqcup_{1\leq j\leq q}C_j))\cap A_r| \geq (\frac{3}{4}+\frac{3}{2}\delta_r - \frac{1}{1000})|A_r|$$

Thus in both cases we have $|((\sqcup_{1\leq i\leq p}B_i)\sqcup(\sqcup_{1\leq j\leq q}C_j))\cap A_r| \geq (\frac{3}{4}+\frac{3}{2}\delta_r - \frac{1}{1000})|A_r|$.

Then $|((\sqcup_{1\leq i\leq p}B_i)\sqcup(\sqcup_{1\leq j\leq q}C_j))\cap A| = \sum_{1\leq r\leq k}|((\sqcup_{1\leq i\leq p}B_i)\sqcup(\sqcup_{1\leq j\leq q}C_j))\cap$

$\cap A_r| \geq \sum_{1\leq r\leq k}(\frac{3}{4}+\frac{3}{2}\delta_r-\frac{1}{1000})|A_r| = \sum_{1\leq r\leq k}(\frac{3}{4}-\frac{1}{1000})|A_r| + \sum_{1\leq r\leq k}\frac{3}{2}\delta_r|A_r| =$

$= (\frac{3}{4}-\frac{1}{1000})|A| + \frac{3}{2}\sum_{1\leq r\leq k} min\{|B\cap A_r|, \frac{1}{4}|A_r| - |B\cap A_r|\} = (\frac{3}{4}-$

$-\frac{1}{1000})|A| + \frac{3}{2}|S| \geq (\frac{3}{4}+\frac{1}{10}-\frac{1}{1000})|A| > (\frac{3}{4}+\frac{1}{100})|A|$. $\square$

The following lemma will cover somewhat different setting and will be used in the sequel. We will call a subset of $\mathbb{Z}$ of cardinality one *a singleton*.

Lemma 5.4. *Let $B_1,\ldots,B_p$ be black triples, $C_1,\ldots,C_q$ be white triples, and $D_1,\ldots,D_r$ be white singletons. Assume all these sets are pairwise disjoint, each of them is a subset of A_i for some $1\leq i\leq k$. (i depending on the set).*

Let $\Phi = \sqcup_{1\leq i\leq p}B_i, \Omega = \Phi\sqcup(\sqcup_{1\leq j\leq q}C_j), \Psi = \sqcup_{1\leq l\leq r}D_l, \Omega' = \Omega\cup\Psi$.

Let also $I = \{s\in\{1,2,\ldots,k\} \mid |\Phi\cap A_s| = 6\}, J = \{s\in\{1,2,\ldots,k\} \mid \Psi\cap A_s \neq \emptyset\}$.

Assume that

(i) for every $s\in\{1,2,\ldots,k\}$, there is no white triple $C\subseteq A_s\backslash\Omega'$.

(ii) for every $s\in J$, $|\Psi\cap A_s|\leq 2$

(iii) for every $s\in J$, $\Phi\cap A_s = \emptyset$.

Then $|\Omega| \geq \frac{3}{4}|A| + |I|$.

Proof. First of all, we observe that condition (iii) implies that $I\cap J = \emptyset$.

For every $1\leq s\leq k$, let

$$X_s = \{i\in\{1,\ldots,p\} \mid B_i\subset A_s\},\ Y_s = \{j\in\{1,\ldots,q\} \mid C_j\subset A_s\}$$

Let also $b_s = |X_s|,\ c_s = |Y_s|$.

Notice that $|\Omega| = \sum_{1\le s\le k} 3(b_s + c_s)$.

We claim that: a) for every $s \in \{1, 2, \dots, k\}$, we have the inequality $3(b_s + c_s) \ge \frac{3}{4}|A_s|$, b) moreover, for every $s \in I$, we have $3(b_s + c_s) \ge \frac{3}{4}|A_s| + 1$.

We will consider two cases:
Case A: Let $s \in \{1, \dots, k\}\backslash J$.

Then condition (i) implies the following
$(i)'$ there is no white triple in $A_s\backslash\Omega$.

First, let us assume that $|(E(A)\backslash\Omega)\cap A_s| \ge 1$ and $|(O(A)\backslash\Omega)\cap A_s| \ge 1$.

If $|(E(A)\backslash\Omega) \cap A_s| \ge 2$ then, because of condition $(i)'$, we have $|(O(A)\backslash\Omega) \cap A_s| = 0$. Then, $s \notin I$, and $|A_s \cap \Omega| \ge \frac{3}{4}|A_s|$. Thus the inequality of part a) holds, and we have nothing to prove for part b).

Now, let us assume $|(E(A)\backslash\Omega)\cap A_s| = 1$. By Lemma 5.2, this implies that $|(O(A)\backslash\Omega)\cap A_s| \le 1$, but since we have assumed $|(O(A)\backslash\Omega)\cap A_s| \ge 1$ we obtain that $|(O(A)\backslash\Omega) \cap A_s| = 1$
Then $|A_s\backslash\Omega| = 1 + 1 = 2$ but this is impossible because the numbers $|\Omega \cap A_s|$ and $|A_s|$ are divisible by 3. Thus we have either $|(E(A)\backslash\Omega) \cap A_s| = 0$ or $|(O(A)\backslash\Omega) \cap A_s| = 0$

For the part a), we will consider the following cases

Case 1: $|(E(A)\backslash\Omega) \cap A_s| = 0$.

Then $b_s + 2c_s = \frac{1}{2}|A_s|$. Then

$$3(b_s+c_s) = (b_s+2c_s)+(2b_s+c_s) \ge (b_s+2c_s)+\frac{1}{2}(b_s+2c_s) = \frac{3}{2}(b_s+2c_s) = \frac{3}{4}|A_s|$$

Thus we proved the claim of part a). For the part b), first, we notice that $|\Phi \cap A_s| = 6$ implies $b_s = 2$. Then,

$$3(b_s+c_s) = (b_s+2c_s)+(2b_s+c_s) = (b_s+2c_s)+\frac{1}{2}(b_s+2c_s)+\frac{3}{2}b_s = \frac{3}{2}(b_s+2c_s)+\frac{3}{2}b_s = \frac{3}{4}|A_s|+3$$

thus the claim of part b) is also proved.

Case 2: $|(O(A)\backslash\Omega) \cap A_s| = 0$.

Then $c_s + 2b_s = \frac{1}{2}|A_s|$. Then

$$3(b_s+c_s) = (c_s+2b_s)+(2c_s+b_s) \ge (c_s+2b_s)+\frac{1}{2}(c_s+2b_s) = \frac{3}{2}(c_s+2b_s) = \frac{3}{4}|A_s|$$

Thus we proved the claim of part a). For the part b), notice that $2b_s \le 4$. Then $c_s \ge \frac{1}{2}|A_s| - 4$. Then $2c_s \ge |A_s| - 8$, but $|A_s| - 8 > \frac{1}{2}|A_s|$. Contradiction, hence for $s \in I$, Case 2 is impossible.

Case B: $s \in J$.

But if $s \in J$, then $s \notin I$ so we have nothing to prove for part b). For part a), since $|\Psi \cap A_s| \leq 2 < 3 \leq \frac{1}{4}|A_s|$, because of conditions (i) and (iii), we obtain that A_s contains exactly $\frac{1}{2}|A_s|$ white triples. So $|A_s \cap \Omega| = \frac{3}{4}|A_s|$. Hence $3(b_s + c_s) \geq \frac{3}{4}|A_s|$.

So, we proved that, for all $s \in \{1, 2, \dots, k\} \backslash I$, (in fact for all $s \in \{1, 2, \dots, k\}$) we have the inequality $3(b_s + c_s) \geq \frac{3}{4}|A_s|$, and for all $s \in I$, we have $3(b_s + c_s) \geq \frac{3}{4}|A_s| + 1$. Adding these inequalities we obtain $|\Omega| \geq \frac{3}{4}|A| + |I|$. $\square$

6. Partitioning the group into η-normal labeled generalized binary trees

Proposition 6.1. *Let* Γ *be a finitely generated group satisfying conditions (A) and (B). Let* $K = \{1, \xi, \xi^{-1}\} \cup \{\eta^i \mid -N \leq i \leq N\}$, $N \geq 24$, $F \subset \Gamma$ *be a finite set,* $F_1 \subseteq Int_K(F)$. *Then there exist disjoint black labeled* η*-normal generalized binary trees* $T_1, T_2, \dots, T_m$ *such that* $F_1 \subseteq \sqcup_{1 \leq i \leq m} S(T_i)$, $|S(T_i) \cap zH| \leq 3, \forall z \in \Gamma, 1 \leq i \leq m$. *Moreover,* $|F_1 \cap \cup_{1 \leq i \leq m} End(T_i)| \leq \frac{1}{4}|F_1|$.

Proof. By definition of the interior, there exists a subset F_0 such that $Int_K F \subseteq F_0 \subseteq F$ and F_0 can be partitioned into pairs of type $(x, x\xi)$. Thus every element $z \in F_0$ has a fixed ξ-*partner* which we will denote by $N_\xi(z)$ and since ξ is fixed we will drop the index denoting it by $N(z)$. Clearly, $N(N(z)) = z, \forall z \in F_0$. In particular, every element in F_1 has a fixed ξ-partner which may not lie in F_1 but at least belongs to F_0.

Let $\{I(x)\}_{x \in X}$ be a collection of horizontal segments of F such that the following conditions are satisfied:
(i) for every two distinct $x_1, x_2 \in X$, $I(x_1) \cap I(x_2) = \emptyset$.
(ii) $F_1 \subseteq \cup_{x \in X} I(x)$.
(iii) for every $x \in X$, $|I(x)|$ is divisible by 12.

We will build the trees $T_1, T_2, \dots$ inductively. For each $1 \leq i \leq m$, we first build the tree S_i and then complete it to T_i through a finite sequence of η-normal black trees $S_1 = S_1^{(0)}, S_1^{(1)}, S_1^{(2)}, \dots$.

For the tree S_1 we start with any white element $x \in F_1$, called *starting element of* T_1, and let $N(x) \in I(x_0), x_0 \in X$. Notice that $N(x)$ is black.

We choose arbitrary two white elements $y, z \in F_1 \cap I(x_0)$ and let $(N(x), y, z,)$ be the root of S_1. (see Figure 4).

We are going to build the vertices η_1, η_2, η_3 which are adjacent to (x, y, z).

We set $\eta_1 = \{x\}$ so η_1 is a singleton vertex. Notice that $N(y), N(z)$ are black. Assume $N(y) \in I(x_1), N(z) \in I(x_2), x_1, x_2 \in X$. Notice that because of condition (B) of Theorem 2.2, $x_1 \neq x_2$. For η_2, if $N(y) \notin F_1$ we let $\eta_2 = \{N(y)\}$ be the end vertex. But if $N(y) \in F_1$ then assuming $N(y) \in I(x_1), x_1 \in X$, we choose arbitrary two white elements $y_1, y_2 \in I(x_1)$ and let $\eta_2 = \{N(y), y_1, y_2\}$. Similarly, for η_3, if $N(z) \notin F_1$, we let $\eta_2 = \{N(y)\}$ be the end vertex. But if $N(z) \in F_1$ then assuming $N(z) \in I(x_2), x_2 \in X$, we choose arbitrary two white elements $z_1, z_2 \in I(x_2)$ and let $\eta_3 = (N(z), z_1, z_2)$.

To build two new triple vertices adjacent to η_2, let $N(y_1) \in I(x_3), N(y_2) \in I(x_4), x_3, x_4 \in X$. Notice that $N(y_1), N(y_2)$ are black. We choose arbitrary two white elements $u_1, u_2 \in I(x_3)$ and let $\theta_1 = (N(y_1), u_1, u_2)$ be one of the new triple vertices adjacent to η_2. Similarly, we choose arbitrary two white elements $v_1, v_2 \in I(x_4)$ and let $\theta_2 = (N(y_2), v_1, v_2)$ be the other new triple vertex adjacent to η_2.

Similarly, we build two new triple vertices adjacent to η_3.

We build vertices of level $n+1$ only after we finish building all vertices of level n.

Because of Lemma 3.16, no two internal vertices of S_1 will lie in the same segment, therefore, all the end vertices of S_1 lie in the boundary $F \backslash F_1$ except the end vertex η_1, but when we extend the tree S_1 to the tree T_1 as described below, all the end vertices of T_1 will lie in the boundary $F \backslash F_1$.

For the *completion process* of $S_1 = S_1^{(0)}$, notice that $\{x\}$ is a singleton vertex of S_1 and x is a white element. Let $x \in I(x_{-1}), x_{-1} \in X$ (since completion is a backward process, we index by negative numbers; although remember that the trees do not have linear order). We choose $u, v \in I(x_{-1})$ such that u is white and v is black. The triple (v, x, u) will be the root of $S_1^{(1)}$ and one of the two branches of $S_1^{(1)}$ is the tree $S_1 = S_1^{(0)}$, and we build the other branch similarly. Notice that the tree $S_1^{(1)}$ is black as well. We continue the process by extending the tree S_1

through the sequence $S_1 = S_1^{(0)}, S_1^{(1)}, S_1^{(2)}, \ldots$ until we reach a tree $S_1^{(k_1)}$ with the root intersecting the boundary $F \backslash F_0$. We define $T_1 = S_1^{(k_1)}$ (See Figure 6.)

Notice that by construction the tree T_1 is black.

For the tree S_2, we pick up a new white element $x' \in I(x_0'), x_0' \in X$, and continue building it similarly. We abide the rule that we do not build any vertex of level $n+1$ of S_2 unless we have finished building all vertices of level n.

At some point in building the tree S_i (or T_i), $1 \leq i \leq m$, we may arrive at a triple vertex $\{a, b, c\} \subset F_1$ of level n such that $N(a)$ belongs to a triple vertex of level $n-1$ of S_i (or T_i) [so a is a central element of the triple (a, b, c)], however, there is no two white elements in $F_1 \cap I(t)$ which do not belong to $\sqcup_{1 \leq j \leq i-1} S(T_j)$ where $t \in X$ is such that $I(t)$ contains $N(b)$. Then we let $\{N(b)\}$ to be the end vertex of S_i (or T_i). So, although all the end vertices of T_1 will lie in the boundary $F \backslash F_1$, this may no longer be true for the trees $T_2, T_3, \ldots$.

In building the trees $S_i, T_i, 1 \leq i \leq m$ we satisfy the following properties.

(i) $S_i, T_i, 1 \leq i \leq m$ are colored ξ-labeled generalized binary trees.

(ii) the trees $T_i, 1 \leq i \leq m$ are disjoint and we do not start building $T_{i+1}, 1 \leq i \leq m-1$ until we have finished building T_i.

(iii) the tree T_i is a completion of the tree S_i through a finite sequence of supernormal labeled generalized binary trees $S_i^{(0)} = S_i, S_i^{(1)}, \ldots, S_i^{(k_i)} = T_i$ such that $S_i^{(j)}$ is one of the 4 branches of the tree $S_i^{(j+1)}$, $\forall j \in \{0, 1, \ldots, k_i - 1\}$ and $r(S_i^{(0)}) \subset F_0, r(S_i^{(k_i)}) = r(T_i) \subset F \backslash F_0$.

(iv) a singleton vertex $\{N(a)\} \in I(x), a \in F_1, x \in X$ of the tree T_i is an end vertex iff either $N(a) \in F \backslash F_1$ or there are no two white elements in $I(x)$ which do not belong to $\sqcup_{1 \leq j \leq i-1} S(T_j)$ where $x \in X$ is such that $I(x)$ contains $N(a)$.

(v) after building the trees $T_1, \ldots, T_{i-1}$, we start a tree T_i at the starting element x such that x is white and $x \in F_1 \backslash \sqcup_{1 \leq j \leq i-1} S(T_j)$.

(vi) We finish building all trees if and only if $F_1 \subseteq \cup_{1 \leq j \leq m} S(T_j)$.

Now, assume we have built all trees $T_1, T_2, \ldots, T_{i-1}$ and all vertices of level n of the tree T_i such that the trees $T_1, \ldots, T_{i-1}$ and the already built subtree of T_i satisfy properties (i)-(vi). Let v_0 be a vertex of level $n \in \mathbb{N} \cup \{0\}$ of the tree T_i such that the vertices of level $n+1$ adjacent to v_0 are not built yet.

We will consider the following cases:

Case 1. v_0 is a singleton vertex.

Then we let v_0 to be an end vertex of T_i, and we pick up arbitrary white element $x \in F_1 \backslash \sqcup_{1\leq j\leq i} S(T_j)$ and start the tree T_{i+1} at x.

Case 2. v_0 is a triple vertex.

In this case, let $v_0 = (a, b, c)$ where $N(a) = p(a)$, i.e. a is the central element of v_0

If $N(b), N(c) \notin F_1$, then we let $\{N(b)\}, \{N(c)\}$ be the end vertices of level $n + 1$ beyond vertex v_0.

If $N(b) \in F_1, N(c) \notin F_1$, then we let the singleton $\{N(c)\}$ be one of the vertices of level $n + 1$ beyond v_0, and we pick arbitrary two black elements b_1, b_2 from the set $I(t) \cap \sqcup_{1\leq j\leq i-1} S(T_j)$ where $t \in X$ is such that $I(t)$ contains $N(b)$ and let $(N(b), b_1, b_2$ be the other vertex. If no such two black elements b_1, b_2 exist then we let the singleton $\{N(b)\}$ be the other vertex next to v_0.

The case $N(b) \notin F_1, N(c) \in F_1$ is symmetric, and finally, in the case, $N(b) \in F_1, N(c) \in F_1$ we build the next vertices similarly.

Notice that none of the properties (i)-(vi) are violated in either of Cases 1-2 hence we may continue by induction.

Notice that, for each $x \in X$, exactly $[I(x)/2]$ elements of $I(x)$ are white (and as many are black), moreover, for each triple v in $I(x)$ (i.e. $S(v) \subseteq I(x)$) exactly two elements of v are white and exactly one element of v is black. Furthermore, notice that all end vertices consists of black elements. Thus the number of singleton vertices in $I(x)$ is at most $\frac{1}{4}|I(x)|$. But since $F_1 \subseteq \cup_{1\leq j\leq m} S(T_j)$, the number of singleton vertices in F_1 is at most $\frac{1}{4}|F_1|$. $\square$

Remark 6.2. Proposition 6.1. is essentially a very special case of Theorem 2.2, however, it is clear that all we need is just a *little push* to accommodate trees better inside F so we can deduce that boundary $F \backslash F_1$ must be big. This is quite in accordance with our intuition that some examples of amenable groups are just *barely amenable*. Of course, this "little push" can be a difficult problem.

Remark 6.3 (Barely amenable groups)**.** We postpone the definition and discussion of these groups to our next publication, but according to our definition, polycyclic groups (consequently, nilpotent groups) and the wreath product $\mathbb{Z}\wr\mathbb{Z}$ are not barely amenable, and the free solvable group of derived length ≥ 2 is. In terms of this paper, barely amenable groups are, roughly, amenable groups satisfying conditions (A) and (C), but we will give more precise definitions in the next publication.

Remark 6.4 (Relaxing the notion of ξ-partner)**.** It is not strictly necessary for the proof that $N_{\xi}(x) \in \{x\xi, x\xi^{-1}\}$. It is fine for the proof if we assume that $N_{\xi}(x) = x\xi^{p(x)}$ where $p(x) \in \mathbb{Z}_{\mathrm{odd}}$, and $max_{x\in F_1} p(x)$ is bounded above in terms of F.

Remark 6.5 (Flows)**.** In the proof of Proposition 6.1, we start a tree and let it *flow* through the set F. It may be difficult at this point but certainly worth of noticing that it would not be better for our purposes to stop the natural flow of the tree and do something artificial to continue it. The effect will not be positive if not negative. The best is to let the tree flow its course. It is also bad to stop the tree abruptly since this increases the percentage of endpoints in the interior of F. Notice that we do have some limited control over the flow, namely, the choice of the side elements (which will have to be white for the black trees) in the set $I(x)$.

We would like to emphasize that, in the proof of Proposition 6.1, during the process of building trees we observe one extra structure on labeled η-normal GBTs, namely, every for every such black (white) tree T one can associate a sequence $S^{(0)}, S^{(1)}, \dots, S^{(k)}$ of special labeled black (white) η-normal GBTs such that the i-the triple on $tail(T)$ is the root of $S^{(i)}, 0 \leq i \leq k$. We will refer to $S^{(0)}$ as the *main part* of the tree T (see Figure 6).

Remark 6.6. In Figure 4, the first three vertices of $tail(T_1)$ are those which contain the elements $N(x), x$ and $N(v)$; and the first four elements of the tail zigzag are $N(x), x, v, N(v)$.

We observe the following

Proposition 6.7. *Let* Γ *be a finitely generated group satisfying conditions* (A) *and* (C)*, and* T *be a labeled black (white)* η*-normal GBT with associated sequence* $S^{(0)}, S^{(1)}, \dots, S^{(k)}$ *of trees in the completion*

process of T where $S^{(k)} = T$ and $y_i, 0 \leq i \leq k$ be the starting element of $S^{(i)}$. Then for all $0 \leq i \leq k,\ x \in S(S^{(i)})\backslash\{y_i\}$ we have $x^{-1}y_i \notin H$.

The above proposition allows to make an extremely important observation:

Remark 6.8 (replacing (B) by (C))**.** We make an important observation that if instead of $(A)+(B)$, we assume $(A)+(C)$, Proposition 6.1. still holds, except we replace the claim "$|S(T_i) \cap zH| \leq 3,\ \forall z \in \Gamma, 1 \leq i \leq m$" by "$|S(r) \cap zH| \leq 3,\ \forall z \in \Gamma, 1 \leq i \leq m$" where r is any level increasing ray. The proof is totally identical.

Remark 6.9. One should notice that just the colored structure (which is provided by condition (A) alone) allows to build a genuine labeled η-normal GBT, although the tree might be very dense, if we do not have condition (B). However, to guarantee the necessary minimal percentage $3/4$ of segments of length $12n, n \in \mathbb{N}$ being filled with internal vertices, we do need condition (C).

The above proposition shows that, if we assume also condition (C), then no element of the tail zigzag (excluding the first two elements) of $S^{(i)}$ can lie on the same horizontal with an element of the main part of $S^{(i)}$ thus we can guarantee that for any segment of length $12n$, at least $9n$ elements of this segment belong to internal vertices of T.

7. Regions suitable for colored zigzags.

In this section, we will be assuming that Γ is a finitely generated group satisfying conditions $(A), (C)$ and (D), $\epsilon > 0$, $K = B_{10000}(1) \subset \Gamma$, F is a connected (K, ϵ)-Folner set.

We will need the following

Definition 7.1. An element $z \in F$ is called *successful* if $h(z\xi^{-p}) > h(z) + 100$. Otherwise, z is called *unsuccessful.*

We will be interested in tiling the group Γ with generalized balanced segments such that each segment has at most one unsuccessful element, moreover, the unsuccessful element is the rightmost element of the segment. Condition (D) immediately allows us to claim the following

Lemma 7.2. *The group Γ can be tiled by a collection of pairwise disjoint balanced generalized segments $\{I(x)\}_{x\in X}$ such that*

(i) all elements of $I(x), x \in X$, except possibly the rightmost element, are successful.

(ii) if for some $x_0 \in X$ the interval $I(x_0)$ has an unsuccessful element z_0 then for all $z \in I(x_0)\backslash\{z_0\}$ the inequality $h(z\xi^{\omega p}) > h(z) + 100$ holds for any $\omega \in \{-1, 1\}$. $\square$

We will fix the tiling $\{I(x)\}_{x\in X}$ satisfying claims of Lemma 7.2 and let $X_0 = \{x \in X \mid I(x) \subset Int_K F\}$, and denote $F_1 = \cup_{x\in X_0} I(x)$. Notice that X_0 is necessarily finite.

Definition 7.3 (Regions)**.** A subset $S \subseteq F_1$ is called a region if there exists $X'_0 \subset X$ such that $S = \sqcup_{x\in X'_0} I(x)$. For any subset $S' \subset F_1$ we will denote the minimal region containing S' by $R(S')$.

Definition 7.4 (Partner assigned regions)**.** A region $S \subseteq F_1$ is called *partner assigned* if there exists a subset $S_0 \subseteq S$ and a function $n : S_0 \to \mathbb{Z}_{\text{odd}}$ such that

$$S \subseteq \sqcup_{x\in S_0}\{x, x\xi^{n(x)}\} \subseteq F$$

We will denote $x\xi^{n(x)} = N(x)$ and $x = N(x\xi^{n(x)})$. We will also denote $exp(x) = n(x), exp(N(x)) = -n(x)$.

Notice that the partners of elements from F_1 do not necessarily lie in F_1 but they always lie in F. In the proof of Theorem 2.2/2.4, we will be assigning partners of elements of F_1 before we start building the trees, so $R(F_1)$ will be a partner assigned region. We will arrange the partner assignment to satisfy certain conditions to enable us to push the rays of the trees to higher levels or at least not to let them come below certain level.

Definition 7.5 (Zigzags respecting partner assignment)**.** Let $S \subseteq F_1$ be a partner assigned region and $Z = (x_1, x_2, \ldots, x_n)$ be a white (black) zigzag in S. We say Z respects partner assignment of S if for every black (white) element $z_i, 1 \le i \le n-1$, we have $z_{i+1} = N(z_i)$.

Definition 7.6 (Regions suitable for colored zigzags)**.** A partner assigned region $S \subseteq F_1$ is called *suitable for black zigzags* if for any black zigzag $Z = (x_1, x_2, \ldots, x_n)$ which respects the tiling $\{I(x)\}_{x\in X}$ and the partner assignment, we have the following conditions satisfied:

(i1) $h(x_{k+14}) > h(x_k) + 100$, for all $k \in \{1, 2, \ldots, n-14\}$.

(i2) $h(x_k) > h(x_1) - 600$, for all $k \in \{1, 2, \ldots, n\}$.

If only condition (i2) is satisfied then the region is called *semi-suitable* for black zigzags.

Similarly, we define regions suitable or semi-suitable for white zigzags.

The following simple lemma will be needed in sequel:

Lemma 7.7. *Let L be any vertical line in $\Gamma, x \in L, L_p(x) = \{x\xi^{pk} \mid k \in \mathbb{Z}\}$. Then there exists $z \in L_p(x)$ such that for all $m > n > 0$, we have $h(z\xi^{pm}) > h(z\xi^{pn}) > h(z)$, and $h(z\xi^{-pm}) > h(z\xi^{-pn}) > h(z)$.* $\square$

Definition 7.8 (p-Sublines, Vertices, Branches)**.** The set $L_p(x)$ will be called *p-subline* of the vertical line L. The element $z \in L_p(x)$ satisfying the conditions of the lemma will be called *the vertex* of the p-subline $L_p(x)$. The set $\{z\eta^n \mid n < 0\}$ is called *the left branch* of L_p and the set $\{z\eta^n \mid n > 0\}$ is called *the right branch* of L_p.

The meaning of the lemma is that the height function on any vertical line (more precisely, on any vertical p-subline) behaves quite strictly in the sense that it strictly descends down to the height of the vertex and then strictly ascends. So, in particular, the height is minimal at the vertex. (see Figure 7.)

Remark 7.9. (see Figure 7) Let $S \subseteq F_1$ be a region. We will consider the following partner assignment: for each $z \in S$, let $x \in X$ be such that $z \in I(x)$. Let also z_0 be the vertex of the p-subline $L_p(z)$.

First, let us assume that z_0 is black. If $z \in \{z_0, z_0\xi^p\} \cap S$ then we choose some element $z' \in L(z) \cap \partial_K F$ of opposite color and set $N(z) = z', N(z') = z$. Then for all black $z \in S \backslash \{z_0, z_0\xi^p\}$ we set $N(z) = z\xi^p$.

If z_0 is white, then for all $z \in \{z_0, z_0\xi^{-p}\} \cap S$ we choose some element $z' \in L(z) \cap \partial_K F$ of opposite color and set $N(z) = z', N(z') = z$. Then for all black $z \in S \backslash \{z_0, z_0\xi^{-p}\}$ we set $N(z) = z\xi^{-p}$.

Remark 7.10. If z_0 is black (white) the pair $\{z_0, z_0\xi^p\}$ (the pair $\{z_0, z_0\xi^{-p}\}$) will be called *a vertex pair.* Thus, for each p-subline of each vertical line, the partners of elements of the vertex pair are partnered in $\partial_K F$. Since on each vertical line we have at most $2p$ elements which belong to vertical pairs, there is enough space in the boundary $\partial_K F$. We will denote the vertex of a p-subline L_p by $v(L_p)$, and the vertex pair of L_p by $vp(L_p)$.

The following proposition will be used in the sequel:

Proposition 7.11. *Let $S \subseteq F_1$ be a partner assigned region where partners of white elements are assigned as in Remark 7.9. Then S is suitable for black zigzags.*

Proof. Let $Z = (z_1, z_2, \ldots, z_n)$ be a black zigzag in S which respects the tiling and partner assignment. Let also $z_i \in I(x_i), x_i \in X, 1 \leq i \leq n$. Without loss of generality we may assume that z_1 is black. (Then $x_2 = x_3, x_4 = x_5$, and so on).

We claim that the zigzag Z has at most one unsuccessful white element. More precisely, we will prove that if z_i is a white unsuccessful element, then z_{i+2r} is successful for all $0 < r \leq \frac{n-1-i}{2}$.

Indeed, let i be the smallest index such that z_i is white and unsuccessful. If $i \in \{n-1, n\}$ we are done. So let $i \leq n-2$. Then $h(z_{i+1}) < h(z_i)$. Notice that z_{i+1} does not belong to any vertex pair (because otherwise the zigzag would terminate at z_{i+1}) therefore by Lemma 7.7, z_{i+1} is unsuccessful hence it is the only unsuccessful element of $I(x_{i+1}) = I(x_{i+2})$. Then the white element z_{i+2} is successful. Then $h(z_{i+3}) > h(z_{i+2}) + 100$ $(\star)$

If $i + 3 \geq n$ then, again, we are done so assume $i \leq n-4$. If z_{i+4} is unsuccessful, by condition (ii) of Lemma 7.2, we have $h(z_{i+3})\xi^{\omega p} > h(z_{i+3}) + 100$ for any $\omega \in \{-1, 1\}$. This contradicts the inequality $(\star)$. Hence z_{i+4} is successful.

By induction on r, we conclude that if $i + 2r \leq n-1$ then z_{i+2r} is successful.

Thus we proved the claim that at most one white element of the zigzag Z is unsuccessful.

Now, let $1 \leq k \leq n-14$. First, let us assume that z_k is black. Then at most one of the elements $z_{k+1}, z_{k+3}, z_{k+5}, z_{k+7}, z_{k+9}, z_{k+11}, z_{k+13}$ is unsuccessful. If none of them is unsuccessful then we get $h(z_{k+14}) > h(z_{k+12}) + 100 > h(z_{k+10}) + 200 > \ldots > h(z_k) + 700 > h(z_k) + 100$. If z_{k+1} is unsuccessful then we obtain $h(z_{k+14}) > h(z_{k+2}) + 600 \geq h(z_k) - 410 - 36 + 600 > h(z_k) + 100$. Similarly, and more generally, if $z_{k+l}, l \in \{1, 3, 5, 7, 9, 11\}$ is unsuccessful, we still obtain $h(z_{k+14}) > h(z_{k+l+1}) + (7 - \frac{l+1}{2})100 > h(z_{k+l-1}) + (7 - \frac{l+1}{2})100 - 410 - 36 > h(z_k) + (7 - \frac{l+1}{2})100 - 410 - 36 + \frac{l-1}{2}100 = h(z_k) - 410 - 36 + 600 > h(z_k) + 100$. If z_{k+13} is unsuccessful, we treat this case similarly.

If z_k is white, then at most one of the elements

$$z_k, z_{k+2}, z_{k+4}, z_{k+6}, z_{k+8}, z_{k+10}, z_{k+12}, z_{k+14}$$

is unsuccessful and we similarly conclude that $h(z_{k+14}) > h(z_k) + 100$.

Finally, the fact that at most one white element of the zigzag Z is unsuccessful immediately implies (i2). $\square$

Similarly, we can define partners of black elements in S to make it suitable for black zigzags, however, notice that it would be problematic to arrange this for both black and white elements of S simultaneously. So, the regions can be arranged to be suitable for black or for white zigzags but not always for both. However, we will be dealing with dividing sets into several regions and making some of them suitable for black zigzags and some of the others suitable for white zigzags. We will discuss this problem in the next section.

Finally, we would like to observe that

Remark 7.12. If $S \subset F_1$ is a region suitable for black zigzags and $Z = (z_1, z_2, \dots, z_n)$ is a white zigzag in S, then
(i) $h(z_{k+14}) < h(z_k) - 100,\ 1 \leq k \leq n - 14$
(ii) $h(z_k) < h(z_1) + 600,\ 1 \leq k \leq n$

8. Intermediate Propositions.

In this section, we will be assuming that Γ is a finitely generated group satisfying conditions $(A), (C)$ and (D), $\epsilon > 0$, $K = B_{10000}(1) \subset \Gamma$, F is a connected (K, ϵ)-Folner set, and $\{I(x)\}_{x \in X}$ is a collection of pairwise disjoint balanced segments tiling Γ and satisfying conditions (i)-(ii) of Lemma 7.2. Let also $X_0 = \{x \in X \mid I(x) \subset Int_K F\}, F_1 = \cup_{x \in X_0} I(x)$.

The plan for the proof of Theorem 2.2: The proof of Proposition 6.1 will play a role of a skeleton for the Proof of Theorem 2.2.

We need to accommodate trees little better to get some positive contribution to the cardinality $|\sqcup_{1 \leq i \leq m} Int(T_i)|$.

For that reason, *very roughly speaking*, we will first partition the set F_1 into certain disjoint union $S_1 \sqcup S_2 \sqcup \dots, S_N \sqcup S_{N+1}$ of finitely many sets such that the sets $S_i, 1 \leq i \leq N$ have large enough interior, and each of $S_i, 2 \leq i \leq N+1$ consists of union of pairwise disjoint balanced segments. (it is useful to think of these sets as concentric annuli, where

S_1 is the outermost annulus, S_2 is the second outermost, and so on, and the very last set S_{n+1} should be imagined as a ball rather than an annulus; see Figure 9).

Let us assume that for all $i \geq 2$, there exists $X_i \subset X_0$ such that $S_i = \sqcup_{x\in X_i} I(x)$.

Let $X_1 = X_0 \setminus \cup_{2\leq i\leq N+1} X_i$. Then $S_1 \subseteq \sqcup_{x\in X_1} I(x)$.

We build pairwise disjoint black trees $T_1^{(1)}, T_2^{(1)}, \ldots, T_{n_1}^{(1)}$ in S_1 such that $|\sqcup_{1\leq j\leq n_1} S(T_j^{(1)})|$ is bigger than certain (small) percentage of $|S_1|$ yet any horizontal segment $I(x), x \in X_1$ contains at most 2 vertices (i.e. at most $2 \times 3 = 6$ elements) of the trees $T_1^{(1)}, T_2^{(1)}, \ldots, T_{n_1}^{(1)}$. After this we switch to white trees $T_{n_1+1}^{(1)}, T_{n_1+2}^{(1)}, \ldots, T_{m_1}^{(1)}$ until all the segments $I(x)$ which contain at least two black vertices (i.e. vertices of the trees $T_1^{(1)}, T_2^{(1)}, \ldots, T_{n_1}^{(1)}$) will now contain also at least two white vertices (i.e. vertices of $T_{n_1+1}^{(1)}, T_{n_1+2}^{(1)}, \ldots, T_{m_1}^{(1)}$).

Because of the choice of ξ-partners, all of these pairwise disjoint trees $T_1^{(1)}, T_2^{(1)}, \ldots, T_{n_1}^{(1)}, T_{n_1+1}^{(1)}, T_{n_1+2}^{(1)}, \ldots, T_{m_1}^{(1)}$ will not intersect the set $S_2 \sqcup S_3 \sqcup \ldots$.

Then we do the same with S_2, i.e. first we cover certain percentage of S_2 with black trees $T_1^{(2)}, T_2^{(2)}, \ldots, T_{n_2}^{(2)}$ such that no segment $I(x), x \in X_2$ contains more than 2 black vertices, and then we switch to white trees $T_{n_2+1}^{(2)}, T_{n_2+2}^{(2)}, \ldots, T_{m_2}^{(2)}$ such that any segment which contains at least two black vertices will contain at least two white vertices.

The trees built in this step do not intersect $S_3 \sqcup S_4 \sqcup \ldots$ but may go into uncovered areas of S_1.

Then we continue the process for S_3, S_4 and so on. The trees $T_1^{(i)}, T_2^{(i)}, \ldots$ $T_{n_i}^{(i)}, T_{n_i+1}^{(i)}, T_{n_i+2}^{(i)}, \ldots, T_{m_i}^{(i)}$ built in step i will be again pairwise disjoint; each of these trees will have their start vertices in S_i; each of these trees will be disjoint with the set $S_{i+1} \sqcup S_{i+2} \sqcup \ldots \sqcup S_N \sqcup S_{N+1}$ but may go into uncovered areas of $S_{i-1} \sqcup S_{i-2} \sqcup \ldots \sqcup S_1$.

We arrange the number N to be big enough (this is possible because we will assume that the group is amenable) that the number of horizontal segments with at least two black and at least two white vertices is bigger than certain percentage of $|F|$.

Then we cover the rest of the set F_1 with white trees $T_{s+1}, \ldots, T_{s+m}$ where $s = m_1 + m_2 + \ldots + m_N$.

The black trees $T_1^{(j)}, \ldots, T_{n_j}^{(j)}, 1 \leq j \leq N$ in the will be called *essential black trees*. The white trees $T_{n_j+1}^{(j)}, \ldots, T_{m_j}^{(j)}, 1 \leq j \leq N$ will be called *essential white trees.* Finally, the trees $T_{s+1}, \ldots, T_m$ will be called *non-essential* trees.

The two propositions of this section materialize the idea (plan) of the proof. First, we need the following

Definition 8.1 (Complete Trees)**.** Let F be a finite subset of Γ.

(i) An η-normal labeled black GBT T of F is called *complete* if for all $v = \{x\} \in Single(T)$ with $x \in I(x_0) \subseteq F_1, x_0 \in X$, if x is black then there are no two white elements in $I(x_0)\backslash S(T)$, and if x is white then there is no $y, z \in I(x_0)\backslash S(T)$ such that y is white and z is black.

(ii) An η-normal labeled white GBT T of F is called *complete* if for all $v = \{x\} \in Single(T)$ with $x \in I(x_0) \subseteq F_1, x_0 \in X$, if x is white then there are no two black elements in $I(x_0)\backslash S(T)$, and if x is black then there is no $y, z \in I(x_0)\backslash S(T)$ such that y is white and z is black.

(iii) A sequence of finitely many labeled η-normal GBTs $T_1, \ldots, T_m$ of F is called complete, if the trees are pairwise disjoint, moreover, for all $1 \leq i \leq m$ and for all $v = \{x\} \in Single(T_i)$ with $x \in I(x_0) \subseteq F_1, x_0 \in X$:

if x is black and T_i is black then there are no two white elements in $I(x_0)\backslash \cup_{1\leq j\leq i} S(T_j)$;

if x is white and T_i is black then there is no $y, z \in I(x_0)\backslash \cup_{1\leq j\leq i} S(T_j)$ such that y is white and z is black;

if x is white and T_i is white then there are no two black elements in $I(x_0)\backslash \cup_{1\leq j\leq i} S(T_j)$;

if x is black and T_i is white then there is no $y, z \in I(x_0)\backslash \cup_{1\leq j\leq i} S(T_j)$ such that y is white and z is black.

The completeness of the sequence $T_1, T_2, \ldots T_m$ simply means that we do not stop the trees unnecessarily. It is important to notice that for a black (white) labeled η-normal GBT T, and end vertex v may consist of a white (black) singleton if and only if v belongs to $tail(T)$.

We need the notion of semi-complete tree as well which one can define for any labeled η-normal GBT, however, we will limit ourselves to black special labeled η-normal trees since this is all we need

Definition 8.2 (Semi-complete Trees)**.** (i) Let T be a special black labeled η-normal GBT. T is called *semi-complete* if $v = \{x\} \in Single(T)$ where $x \in I(x_0) \subseteq F_1, x_0 \in X$ and x is black, then there are no two white elements in $I(x_0)\backslash S(T)$.

(ii) Let $T_1, \ldots, T_m$ of F be a finite sequence of labeled η-normal GBTs such that if $T_i, 1 \leq i \leq m$ is black then it is special. The sequence is called semi-complete, if the trees are pairwise disjoint, moreover, for all $1 \leq i \leq m$ and for all $v = \{x\} \in Single(T_i)$ with $x \in I(x_0) \subseteq F_1, x_0 \in X$:

if x is black and T_i is black then there is no two white elements in $I(x_0) \setminus \cup_{1 \leq j \leq i} S(T_j)$;

if x is white and T_i is white then there is no two black elements in $I(x_0) \setminus \cup_{1 \leq j \leq i} S(T_j)$;

if x is black and T_i is white then there is no $y, z \in I(x_0) \setminus \cup_{1 \leq j \leq i} S(T_j)$ such that y is white and z is black.

Notice that in part (ii) of the definition, we do ignore the case "if x is white and T_i is black". Indeed this case occurs only if x is the starting element of T_i.

Proposition 8.3. *Let $X_0' \subseteq X_0$ and $T_1, T_2, \ldots, T_m$ be semi-complete labeled η-normal special black trees of F such that*

(R1) $S(T_i) \cap S(T_j) = \emptyset,\ 1 \leq i \neq j \leq m$.

(R2) for every $x \in X_0$, the balanced generalized segment $I(x)$ contains at most two white elements which are starting elements of black special tree T_i for some $1 \leq i \leq m$; and if $I(x)$ contains at least one such white element, then $I(x) \cap \sqcup_{1 \leq i \leq m} Int(T_i) = \emptyset$.

(R3) for all $x \in X_0'$, $|\sqcup_{1 \leq i \leq m} S(T_i) \cap I(x)| = 6$.

Then $|\partial_K F| \geq \frac{1}{3}|X_0'|$.

Proof. We build labeled η-normal white trees $T_{m+1}, \ldots, T_n$ of F such that the following conditions are satisfied:

(c1) the trees $T_1, \ldots, T_n$ are all pairwise disjoint (i.e. $S(T_i) \cap S(T_j) = \emptyset$ for all $1 \leq i \neq j \leq n$).

(c2) for every $x \in X_0$, the set $I(x) \setminus \sqcup_{1 \leq i \leq n} S(T_i)$ does not contain any white triple.

(c3) the sequence $T_1, T_2, \ldots, T_n$ is semi-complete.

Now, let

$$V = \{v \mid v \in \sqcup_{1 \leq i \leq n} Int(T_i), v \textit{ is a black vertex}\} = \{v_1, v_2, \ldots, v_p\},$$

$$U = \{v \mid v \in \sqcup_{1 \leq i \leq n} Int(T_i), v \textit{ is a white vertex}\} = \{u_1, u_2, \ldots, u_q\}.$$

$$W = \{w \mid w \in \sqcup_{1 \leq i \leq m} End(T_i), w = start(T_i), 1 \leq i \leq m\} = \{w_1, w_2, \ldots, w_l\}.$$

Also, for every $x \in X_0$, let

$$B(x) = Card(\{v \mid v \in V, S(v) \subseteq I(x)\}),\ W(x) = Card\left(\{v \mid v \in U, S(v) \subseteq I(x)\}\right)$$

Then, since every internal vertex is a triple,

$$|I(x) \cap \sqcup_{1\le i\le m} Int(T_i)| = 3(B(x) + W(x)),\ \forall x \in X_0$$

We are going to show that

$$\sum_{x\in X_0} (3B(x) + 3W(x)) > \frac{3}{4}|F_1| + |X_0|\ (\star)$$

Let $|X_0| = k, f : X_0 \to \{1, 2, \dots, k\}$ be a bijective map.

Then we denote $I(x) = A_{f(x)},\ B_i = S(v_i), 1 \le i \le p,\ C_j = S(u_j), 1 \le j \le q,\ D_l = S(w_l), 1 \le l \le r, X_0' = I$. Let $A = \sqcup_{x\in X_0} A_{f(x)}$.

Because of condition $(c1)$, the sets $B_i, 1 \le i \le p,\ C_j, 1 \le j \le q,\ D_l, 1 \le l \le r$ are pairwise disjoint. By definition, the sets $B_i, 1 \le i \le p$ are black triples, the sets $C_j, 1 \le j \le q$ are white triples, and the sets $D_l, 1 \le l \le r$ are white singletons.

Let $\Phi = \sqcup_{1\le i\le p} B_i, \Psi = \sqcup_{1\le l\le r} D_l, \Omega = (\sqcup_{1\le i\le p} B_i) \sqcup (\sqcup_{1\le j\le q} C_j)$.

[So $\Omega = \sqcup_{1\le i\le n} Int(T_i)$; in other symbols, $\Omega = (\sqcup_{1\le i\le p} S(v_i)) \sqcup (\sqcup_{1\le j\le q} S(u_j))$].

Because of condition $(c2)$, for each $I(x), x \in X_0$, condition (i) of Lemma 5.4. is satisfied. By conditions $(R2)$ and$(R1)$, the conditions (ii) and (iii) of Lemma 5.4. are also satisfied.

Then, by Lemma 5.4, we obtain $| \sqcup_{y\in Y} I(y) \cap \Omega| \ge \frac{3}{4}|F_1| + |X_0'|$ thus we proved inequality $(\star)$.

Thus $|\Omega| \ge \frac{3}{4}(|F_1| + \frac{4}{3}|X_0'|)$. Then by Lemma 3.10, we have the inequality

$$| \sqcup_{1\le i\le n} End(T_i)| \ge \frac{1}{4}(|F_1| + \frac{4}{3}|X_0'|) = \frac{1}{4}|F_1| + \frac{1}{3}|X_0'|$$

But since $|\sqcup_{1\le i\le n} End(T_i) \cap F_1| \le \frac{1}{4}|F_1|$, (see claim a) in the proof of Lemma 5.4.) we obtain the inequality $|\sqcup_{1\le i\le n} End(T_i) \cap \partial_K F| \ge \frac{1}{3}|X_0'|$. Then $|\partial_K F| \ge \frac{1}{3}|X_0'|$. $\square$

Before stating the next proposition, let us remark that the main difficulty in the plan is the problem that when we try to cover the set F (or $S_i, 1 \le i \le N$) we try to cover as much of it as possible abiding the condition that no, say, three vertices of no three (or less than three)

black trees are on the same horizontal and close to each other. Once we see this is no longer possible, we start using the white trees. The problem is how to maximize the percentage of black trees.

More precisely, suppose we start covering the set F (or S_i) with labeled η-normal black trees $T_1, \ldots, T_k$ such that no segment $I(x) \subset F_1$ intersects more than 2 vertices. Let $M(k)$ denotes the number of segments $I(x)$ which intersect at least two vertices of $T_1, \ldots, T_k$. If k is small enough then we can arrange $M(k)$ to increase as k increases. But if k is very big, it might be impossible to make sure that $M(k)$ increases, abiding at the same time the condition that no segment $I(x)$ intersects more than 2 vertices, i.e. we cannot add a new tree to increase $M(k)$ at the same time satisfying the required condition. The question is how maximal can $M(k)$ get? One can relax the condition "no segment intersects more than 2 vertices" by asking that at least the number of such segments is big enough. Conditions (A) and (C) indeed allow to claim that such number is comparable to the size of the boundary of F (we will discuss this issue in our next publication). However, we need more than that therefore we need more subtle control over the behaviour of the trees.

In the proof of Theorem 2.2/2.4 we will divide sets into regions and arrange the partner assignment so that some of the regions are suitable for black zigzags and some of the others are suitable for white zigzags. The following proposition will be needed for that purpose.

Proposition 8.4. *Let $c_1, c_2, \ldots, c_{4n-1}, c_{4n} = 0$ be a decreasing sequence of nonnegative integers, such that $c_1 \leq max\{h(z) \mid z \in F_1\}$ and $c_i - c_{i+1} > 1000, 1 \leq i \leq 4n-1$. Let also $M_i = \{z \in F_1 \mid c_{i-1} \leq h(z) \leq c_i\}, 1 \leq i \leq 4n$. Then one can assign ξ-partners to elements of F_1 such that the regions $R(M_{4i+1}), 0 \leq i \leq n-1$ become suitable for black zigzags and the regions $R(M_{4i+3}), 0 \leq i \leq n-1$ become suitable for white zigzags.*

Proof. Let $R = (\sqcup_{1 \leq i \leq n} R(M_{4i+1})) \sqcup (\sqcup_{1 \leq i \leq n} R(M_{4i+3}))$. For every vertical p-subline L_p such that $L_p \cap R \neq \emptyset$, let $v(L_p) \in M_{4m+1} \cup M_{4m+2} \cup M_{4m+3} \cup M_{4m+4}$. Then, for each $i \in [1, m) \cap \mathbb{Z}$ we choose x_i, y_i, z_i, t_i such that

(i) x_i, y_i belong to the left branch of L_p and z_i, t_i belong to the right branch of L_p.

(ii) $x_i, z_i \in M_{4i+2} \backslash R,\ y_i, t_i \in M_{4i+4} \backslash R$.

(iii) x_i, t_i are black, y_i, z_i are white. (see Figure 8.)

We consider the following two cases:

Case 1. $v(L_p) \in M_{4m+1} \cup (M_{4m+2} \backslash R)$.

Then we set $N(x) \in \partial_K F$ if x belongs to vertex pair of L_p; and $N(x_i) = y_i, N(z_i) = t_i, \forall i \in [1, m) \cap \mathbb{Z}$.

Moreover, if $x \notin \cup_{1 \leq i < m} \{x_i, y_i, z_i, t_i\} \cup vp(L_p)$ then we simply set $N(x) = x\xi^{-p}$ if x is white and $x \in \sqcup_{1 \leq i < m} (int(x_i, y_{i-1}) \sqcup int(z_i, t_{i-1}) \sqcup int(t_{m-1}, y_{m-1}))$; and we set $N(x) = x\xi^p$ if x is white and $x \in \sqcup_{1 \leq i < m} (int(y_i, x_i) \sqcup int(t_i, z_i))$.

Case 2. $v(L_p) \in M_{4m+3} \cup (M_{4m+4} \backslash R)$.

In this case, we also choose x_m, z_m such that x_m is black, belongs to the left branch, and y_m is white, belongs to the right branch. We set $x \in \partial_K F$ if $x \in vp(L_p), N(x_i) = y_i, N(z_i) = N(t_i)$, for all $1 \leq i < m$; we also set $N(x_m) = z_m$.

If $x \notin \cup_{1 \leq i < m} \{x_i, y_i, z_i, t_i\} \cup vp(L_p) \cup \{x_m, z_m\}$ then we let $N(x) = x\xi^{-p}$ if x is white and $x \in \sqcup_{1 \leq i \leq m} ((int(x_i, y_{i-1}) \sqcup int(z_i, t_{i-1}))$, and we let $N(x) = x\xi^p$ if x is white and $x \in \sqcup_{1 \leq i < m} (int(y_i, x_i) \sqcup int(t_i, z_i)) \sqcup int(x_m, z_m)$. $\square$

We would like to point out the following useful corollary

Remark 8.5. Under the assumptions of Proposition 8.4, if T is a labeled η-normal white GBT with the root at $R(M_{4i+1})$ then $S(T) \cap M_j = \emptyset$ for all $j \geq 4i + 4$. In other words, the tree does not go below the level c_{4i+4}. This is because the tail zigzag of T will either remain in C_{4i+1} or go outwards (higher level), and since the set $S(T)$ consists of the union of tail zigzag (which is black) and some number of white zigzags, the white zigzags will not go below the level of C_{4i+3}

9. The Proof of Theorem 2.4.

Let Γ be a finitely generated amenable group satisfying conditions $(A), (C)$ and (D), $\epsilon > 0$, $K = B_{10000}(1) \subset \Gamma$, F be a connected (K, ϵ)-Folner set, $\{I(x)\}_{x \in X}$ be a collection of pairwise disjoint balanced segments tiling Γ and satisfying conditions (i)-(ii) of Lemma 7.2. Let also $X_0 = \{x \in X \mid I(x) \subset Int_K F\}, F_1 = \cup_{x \in X_0} I(x)$. Finally, for $x, y \in \Gamma$, let $d(x, y)$ denotes the distance in the Cayley metric of Γ with respect to generating set $\{\xi, \eta, \xi^{-1}, \eta^{-1}\}$.

We choose $\epsilon > 0$ small enough that there exists a natural number N, a finite sequence $(h_1, h_2, \ldots, h_N)$ of natural numbers, and a subset $X_0' \subset X_0$ satisfying the following conditions:

(i) $h_i - h_{i+1} > 16000$, forall $1 \leq i \leq N-1$.

(ii) for all $z \in F_1$, $h(z) < h_1 + 3000$.

(iii) for all $x \in X_0'$ there exists $i \in \{1, 2, \ldots, N\}$, and $z \in I(x)$ such that $h(z) \in [h_i - 500, h_i + 500]$.

(iv) for all two distinct $x', x'' \in X_0'$ if there exists $z' \in I(x'), z'' \in I(x'')$ such that $|h(z') - h(z'')| \leq 1000$ then we have $d(z', z'') > 10^7$.

(v) $|X_0'| > 100|\partial_K F|$.

Let $X_i = \{x \in X_0' \mid \exists z \in I(x), h(z) \in [h_i - 500, h_i + 500]$, $Y_i = \cup_{x \in X_i} I(x)$, $1 \leq i \leq N$.

Because of condition (i), $Y_i \cap Y_j = \emptyset$, for all $1 \leq i \neq j \leq N$. We also let $X_i = \{x_1^{(i)}, x_2^{(i)}, \ldots, x_{k_i}^{(i)}\}, 1 \leq i \leq N$. (so we choose any ordering of the set X_i).

Let $A_i = \{z \in F_1 \mid h(z) \in (h_i - 4000, h_i + 4000]\}$,

$B_i = \{z \in F_1 \mid h(z) \in (h_i - 8000, h_i - 4000]\}$,

$C_i = \{z \in F_1 \mid h(z) \in (h_i - 12000, h_i - 8000]\}$,

$D_i = \{z \in F_1 \mid h(z) \in (h_{i+1} + 4000, h_{i+1} - 12000]\}$,

$U(A_i) = \{z \in F_1 \mid h(z) \in (h_i + 3000, h_i + 4000]\}$,

$U(C_i) = \{z \in F_1 \mid h(z) \in (h_i - 9000, h_i - 8000]\}$,

$M(C_i) = \{z \in F_1 \mid h(z) \in (h_i - 11000, h_i - 10000]\}$

for all $1 \leq i \leq N$. We also let, for all $1 \leq i \leq N$, $S_i = (A_i \backslash U(A_i) \cup B_i \cup C_i \cup D_i \cup U(A_{i+1})$ by letting $U(A_{N+1}) = \emptyset$.

Because of Proposition 8.4, we can define ξ-partners such that the region $R(A_i)$ will be suitable for white zigzags, and the region $R(C_i)$ will be suitable for black zigzags.

Notice that for all $x \in X_0'$ and for all $z \in I(x)$, we have $z \in \sqcup_{1 \leq i \leq N} A_i$.

We choose any black triple in the segment $I(x_1^{(1)})$ and build a special black labeled semi-complete η-normal GBT $T_1^{(1)}$ with the root at this triple.

Notice that since $R(A_1)$ is suitable for white zigzags, and $R(C_1)$ is suitable for black zigzags, and moreover, $T_1^{(1)}$ is special, we have $S(T_1^{(1)}) \subseteq A_1 \cup B_1 \cup U(C_1)$.

Then we build another special black labeled semi-complete η-normal GBT $T_2^{(1)}$ with the root at a black triple in $I(y_1^{(1)})\backslash S(T_1^{(1)})$. Notice that, again, $S(T_2^{(1)}) \subseteq A_1 \cup B_1 \cup U(C_1)$; moreover, because of condition (C), the intersection $(S(T_1^{(1)}) \sqcup S(T_2^{(1)})) \cap I(y_1^{(1)})$ consists of the disjoint union of two black triples, namely, the roots of $T_1^{(1)}$ and $T_2^{(1)}$.

Then we choose a black triple in the segment $I(x_2^{(1)})$ and build a special black labeled semi-complete η-normal GBT $T_3^{(1)}$. Then, choosing another black triple in $I(x_2^{(1)})$, disjoint with $root(T_3^{(1)})$, we build a fourth special black labeled semi-complete η-normal GBT $T_4^{(1)}$.

Notice that, again, $S(T_i^{(1)}) \subseteq A_1 \cup B_1 \cup U(C_1)$ for $3 \leq i \leq 4$. Moreover, the intersection $(S(T_1^{(1)}) \sqcup S(T_2^{(1)})) \cap I(y_2^{(1)})$ consists of the union of two black triples.

It is important to notice that, because of condition (iv) and Remark 7.12, the trees $T_1^{(1)}, T_2^{(1)}$ do not intersect the segment $I(x_2^{(1)})$, and the trees $T_3^{(1)}, T_4^{(1)}$ do not intersect the segment $I(x_1^{(1)})$.

As in the proof of Theorem 6.1, the built trees are disjoint, i.e. $S(T_i^{(1)}) \cap S(T_j^{(1)}) = \emptyset$ for all $1 \leq i \neq j \leq 4$.

By continuing the process, for each $x_i^{(1)}, 1 \leq i \leq k_1$, we build a pair of black labeled semi-complete η-normal GBTs $T_{2i-1}^{(1)}, T_{2i}^{(1)}$ such that the following conditions are satisfied:

(s_1) $S(T_i^{(1)}) \cap S(T_j^{(1)}) = \emptyset$ for all $1 \leq i \neq j \leq 2k_1$

(t_1) the roots of the trees $T_{2i-1}^{(1)}, T_{2i}^{(1)}, 1 \leq i \leq k_1$ are disjoint and belong to the balanced segment $I(x_i^{(1)})$.

(u_1) the trees $T_{2i-1}^{(1)}, T_{2i}^{(1)}, 1 \leq i \leq k_1$ are special.

(v_1) $S(T_i^{(1)}) \subseteq A_1 \cup B_1 \cup U(C_1)$ for all $1 \leq i \leq 2k_1$

Then we start building white labeled complete η-normal GBTs $T_{2k_1+1}^{(1)}, \ldots, T_{m_1}^{(1)}$ which satisfy the following conditions:

(a_1) $S(T_i^{(1)}) \cap S(T_j^{(1)}) = \emptyset$, for all $1 \leq i \neq j \leq m_1$.

(b_1) $S(T_i^{(1)}) \subseteq A_1 \cup B_1 \cup C_1 \cup D_1 \cup U(A_2)$ for all $2k_1 + 1 \leq i \leq m_1$

(c_1) for each $1 \leq i \leq k_1$, the segment $I(x_i^{(1)})$ contains at least two white triples.

(d_1) for any $x \in X$, there is no white triple in $(I(x) \cap M(C_1)) \backslash \sqcup_{1 \leq i \leq m_1} S(T_i^{(1)})$

Notice that condition (b_1) is guaranteed because the region $R(C_1)$ is suitable for black zigzags and region $R(U(A_2))$ is suitable for white

zigzags. Condition (d_1) can be also guaranteed because if indeed there is a white triple in $(I(x) \cap M(C_1)) \setminus \sqcup_{1 \leq i \leq m_1} S(T_i^{(1)})$ then we could start building a new white tree with the root in that triple.

Notice that because of conditions (v_1) and (b_1), we have $\sqcup_{1 \leq i \leq m_1} S(T_i^{(1)}) \subseteq (A_1 \setminus U(A_1)) \cup B_1 \cup C_1 \cup D_1 \cup U(A_2)$. In other words, $\sqcup_{1 \leq i \leq m_1} S(T_i^{(1)}) \subseteq S_1$.

Then we go to S_2 and first start building the black trees $T_i^{(2)}, 1 \leq i \leq 2k_2$ and then white trees $T_i^{(2)}, 2k_2 + 1 \leq i \leq m_2$.

Generally, in the q-th step, $1 \leq q \leq N$, we build the black trees $T_i^{(q)}, 1 \leq i \leq 2k_q$ and then white trees $T_i^{(q)}, 2k_q + 1 \leq i \leq m_q$ which satisfy the following conditions:

(s_q) $S(T_i^{(q)}) \cap S(T_j^{(q)}) = \emptyset$ for all $1 \leq i \neq j \leq 2k_q$

(t_q) the roots of the trees $T_{2i-1}^{(q)}, T_{2i}^{(q)}, 1 \leq i \leq k_q$ are disjoint black triples in $I(x_i^{(q)})$.

(u_q) the trees $T_{2i-1}^{(q)}, T_{2i}^{(q)}, 1 \leq i \leq k_1$ are special.

(v_q) $S(T_i^{(q)}) \subseteq A_q \cup B_q \cup U(C_q)$ for all $1 \leq i \leq 2k_q$

and

(a_q) $S(T_i^{(q_1)}) \cap S(T_j^{(q_2)}) = \emptyset$, for all $q_1, q_2 \in \{1, \ldots, q\}, 1 \leq i \neq j \leq m_q$.

(b_q) $S(T_i^{(q)}) \subseteq A_q \cup B_q \cup C_q \cup D_q \cup U(A_{q+1})$ for all $2k_q + 1 \leq i \leq m_1$.

(c_q) for each $1 \leq i \leq k_q$, the segment $I(x_i^{(q)})$ contains at least two white triples.

(d_q) for any $x \in X$, there is no white triple in $(I(x) \cap (M(C_q)) \setminus \sqcup_{1 \leq i \leq m_q} S(T_i^{(q)})$.

Then the rest of F_1 will be covered by white labeled complete η-normal GBTs $T_{s+1}, \ldots, T_m$ where $s = m_1 + m_2 + \ldots + m_N$ until to the point that no segment $I(x) \subset F_1, x \in X$ contains a white triple which is uncovered, i.e. there is no white triple in $I(x) \setminus \sqcup_{1 \leq i \leq m} S(T_i)$.

Now, let $s_i^{(q)}$ denotes the starting element of the tree $T_i^{(q)}$, $1 \leq i \leq m_q$, and let $y_i^{(q)} \in X$ be such that $s_i^{(q)} \in I(y_i^{(q)})$. Notice that because of condition (C), for all $1 \leq i \leq k_q$ we have

$$[S(T_{2i-1}^{(q)}) \sqcup S(T_{2i}^{(q)})] \cap [I(y_{2i-1}^{(q)}) \cup I(y_{2i}^{(q)})] = \{s_{2i-1}^{(q)}, s_{2i}^{(q)}\}$$

Moreover, because of conditions (iv) and Remark 7.12, for all $1 < j \leq k_q, j \neq i$, $[S(T_{2j-1}^{(q)}) \sqcup S(T_{2j}^{(q)})] \cap [I(y_{2i-1}^{(q)}) \cup I(y_{2i}^{(q)})] = \emptyset$

Furthermore, because of conditions $(b_q), (v_q), 1 \leq q \leq N$, for all $q < q' \leq N,\ 1 \leq j \leq 2k_{q'}$, we have

$$[S(T^{(q')}_{2j-1}) \sqcup S(T^{(q')}_{2j})] \cap [I(y^{(q)}_{2i-1}) \cup I(y^{(q)}_{2i})] = \emptyset$$

Thus, for any $1 \leq q \leq N, 1 \leq i \leq k_q$, the segments $I(y^{(q)}_{2i-1})$ and $I(y^{(q)}_{2i})$ do not contain any black triple and may contain at most two white elements which do not belong to any internal vertex. Thus condition $(R2)$ of Proposition 8.3. is satisfied.

To verify condition $(R3)$, (we identify the X'_0 in this proof with X'_0 in the statement of Proposition 8.3.), notice that because of condition (C), for all $1 \leq i \leq k_q$ we have

$$[S(T^{(q)}_{2i-1}) \sqcup S(T^{(q)}_{2i})] \cap I(x^{(q)}_i) = root(T^{(q)}_{2i-1}) \sqcup root(T^{(q)}_{2i})$$

Moreover, because of conditions (iv) and Remark 7.12, for all $1 < j \leq k_q, j \neq i,\ [S(T^{(q)}_{2j-1}) \sqcup S(T^{(q)}_{2j})] \cap I(x^{(q)}_i = \emptyset$.

Finally, because of conditions $(b_q), (v_q), 1 \leq q \leq N$, for all $q < q' \leq N,\ 1 \leq j \leq 2k_{q'}$, we have $[S(T^{(q')}_{2j-1}) \sqcup S(T^{(q')}_{2j})] \cap I(x^{(q)}_i) = \emptyset$.

Thus we verified condition $(R3)$ as well. Condition $(R1)$ is satisfied because of (a_q).

Then, by Proposition 7.3 and by condition (v), we have $|\partial_K F| > \frac{1}{3}|\bar{X}'_0| > \frac{100}{3}|\partial_K F|$. We obtained contradiction. $\square$

Part 2: Application to R.Thompson's group F.

In 1965 Richard Thompson introduced a remarkable infinite group $\mathbf{F}$ that has two standard presentations: a finite presentation with two generators and two relations, and an infinite presentation that is more symmetric.

$$\mathbf{F} \cong \langle A, B \mid [AB^{-1}, A^{-1}BA] = 1, [AB^{-1}, A^{-2}BA^2] = 1\rangle$$
$$\cong \langle X_0, X_1, X_2, \dots \mid X_n X_m = X_m X_{n+1},\ \forall\ n > m\rangle$$

Basic properties of $\mathbf{F}$ can be found in [CFP]. The standard isomorphism between the two presentations of $\mathbf{F}$ identifies A with X_0 and B with X_1. For convenience, let $\mathbb{A} = \{A, B\} = \{X_0, X_1\}$, let $\mathbb{X} = \{X_0, X_1, X_2, \dots\}$ and let $\textsc{Free}(\mathbb{A})$ and $\textsc{Free}(\mathbb{X})$ denote the free groups of rank 2 and of countably infinite rank with bases $\mathbb{A}$ and $\mathbb{X}$, respectively.

10. Special classes of words

Recall the following basic fact about elements in free groups.

Proposition 10.1 (Syllable normal form)**.** *There is a natural one-to-one correspondence between non-trivial elements in* $\text{Free}(\mathbb{X})$ *and words* $X_{n_1}^{e_1} X_{n_2}^{e_2} \cdots X_{n_k}^{e_k}$ *with nonzero integer exponents and distinct adjacent subscripts.*

A word of the form described in Proposition 10.1 is called the *syllable normal form* of the corresponding element in $\text{Free}(\mathbb{X})$. The terminology refers to the language metaphor under which an element of $\mathbb{X}$ is a *letter*, a finite string of letters and their formal inverses is a *word*, and a maximal subword of the form X_n^e is a *syllable*. The next definition makes it easy to describe interesting subsets of elements in $\text{Free}(\mathbb{X})$ based on the properties of their syllable normal forms.

Definition 10.2 ((E, A)-words)**.** Let $W = X_{n_1}^{e_1} X_{n_2}^{e_2} \ldots X_{n_k}^{e_k}$ be a word over $\mathbb{X}$ in syllable normal form and let A and E be subsets of $\mathbb{Z}^*$. We call W an (E, A)-*word* if every exponent e_i belongs to E and the difference in adjacent subscripts (i.e. $n_{i+1} - n_i$) is always an element in A. For example, every word in syllable normal form is a $(\mathbb{Z}^*, \mathbb{Z}^*)$-word, and W is a $(\mathbb{Z}_{\text{odd}}, \mathbb{Z}_{\text{odd}})$-word iff every exponent is odd and the parity of the subscripts strictly alternates. The symbols $\mathbb{Z}^*$ and $\mathbb{Z}_{\text{odd}}$ refer, of course, to the nonzero and the odd integers, respectively.

Definition 10.3 (Reduced words and odd words)**.** The situations where $E = A = \mathbb{Z}^*$ or $E = A = \mathbb{Z}_{\text{odd}}$ are of sufficient interest to merit their own names. When W is a $(\mathbb{Z}^*, \mathbb{Z}^*)$-word, (which is true precisely when W is in syllable normal form), we say it is a *reduced word*, and when W is a $(\mathbb{Z}_{\text{odd}}, \mathbb{Z}_{\text{odd}})$-word, we say it is an *odd word*. As should be clear from the definitions, when $E \subset E'$ and $A \subset A'$, every (E, A)-word is an (E', A')-word. Thus every odd word is reduced and represents a well-defined element of $\text{Free}(\mathbb{X})$. Also note that the inverse of an (E, A)-word is an (E, A)-word so long as E and A are symmetric sets of integers; in general W^{-1} is merely a $(-E, -A)$-word. Thus the inverse of a reduced word is reduced and the inverse of an odd word is odd.

The main result in this section, Theorem 10.12, is that none of the elements in the kernel of the homomorphism $\text{Free}(\mathbb{X}) \to \mathbf{F}$ are represented by odd words. Said differently, every odd word represents a non-trivial element of $\text{Free}(\mathbb{X})$ whose image in $\mathbf{F}$ remains non-trivial. To establish non-triviality, we use the following result on normal forms for elements in Thompson's group $\mathbf{F}$. For a proof of this result see [CFP].

Theorem 10.4 (Thompson normal forms)**.** *Every element in* $\mathbf{F}$ *can be represented by a word* W *in the form* $X_{n_0}^{e_0} X_{n_1}^{e_1} \ldots X_{n_k}^{e_k} X_{m_l}^{-f_l} \ldots X_{m_0}^{-f_0}$ *where the e's and f's are positive integers and the n's and m's are non-negative integers satisfying* $n_0 < n_1 < \ldots < n_k$ *and* $m_0 < m_1 < \ldots < m_l$*. If we assume, in addition, that whenever both* X_n *and* X_n^{-1} *occur, so does either* X_{n+1} *or* X_{n+1}^{-1}*, then this form is unique and called the* Thompson normal form *of this element.*

In order to cleanly describe the rewriting process used to convert an arbitrary reduced word into its equivalent Thompson normal form (and to explain the reason for the final restrictions), it is useful to introduce some additional terminology.

Definition 10.5 (Shift map)**.** Let S denote the map that systematically increments subscripts by one. For example, if $W = X_{n_1}^{e_1} X_{n_2}^{e_2} \ldots X_{n_k}^{e_k}$, then $S(W)$ is the word $X_{n_1+1}^{e_1} X_{n_2+1}^{e_2} \ldots X_{n_k+1}^{e_k}$. More generally, for each $i \in \mathbb{N}$, let $S^i(W)$ denote i applications of the shift map to W. Thus, $S^i(W) = X_{n_1+i}^{e_1} X_{n_2+i}^{e_2} \ldots X_{n_k+i}^{e_k}$. Note that this process can also be reversed, a process we call *down shifting*, so long as all of the resulting subscripts remain non-negative. Also note, that a shift of an odd word, up or down, remains an odd word.

Remark 10.6 (Rewriting words)**.** Using the shift notation, the defining relations for $\mathbf{F}$ can be rewritten as follows: for all $n > m$, $X_n X_m = X_m S(X_n)$. More generally, let $W = X_{n_1}^{e_1} X_{n_2}^{e_2} \ldots X_{n_k}^{e_k}$ be a reduced word and let $\min(W)$ denote the smallest subscript that occurs in W, i.e. $\min(W) = \min\{n_1, n_2, \ldots, n_k\}$. It is easy to show that for all words W with $\min(W) > m$, $W X_m = X_m S(W)$. Similarly, $X_m^{-1} W = S(W) X_m^{-1}$. After iteration, for all positive integers e, $W X_m^e = X_m^e S^e(W)$ and $X_m^{-e} W = S^e(W) X_m^{-e}$. The reason for the extra restriction in the statement of Theorem 10.4 should now be clear. If W is in Thompson normal form with $\min(W) > m$, then $X_m S(W) X_m^{-1}$ is an equivalent word that satisfies all the conditions except the final restriction.

Definition 10.7 (Core of a word)**.** Let W be a reduced word with $\min(W) = m$. By highlighting those syllables that achieve this minimum, W can be viewed as having the following form:

$$W = W_0 X_m^{e_1} W_1 X_m^{e_2} \ldots X_m^{e_l} W_l$$

where the e's are nonzero integers, each word W_i is a reduced word with $\min(W_i) > m$, always allowing for the possibility that the first and last words, W_0 and W_k, might be the empty word.

We begin the process of converting W into its Thompson normal form by using the rewriting rules described above to shift each syllable

$X_m^{e_i}$ with e_i positive to the extreme left and each such syllable with e_i negative to the extreme right. This can always be done at the cost of increasing the subscripts in the subwords W_i. If we let *pos* and *neg* denote the sum of the positive and negative e's, respectively, then W is equivalent in $\mathbf{F}$ to a word of the form $W' = X_m^{pos} W_0' W_1' \dots W_l' X_m^{neg}$ with W_i' is an appropriate upward shift of the word W_i. The appropriate shift in this case is the sum of the positive X_m exponents in W to the right of W_i plus the absolute value of the sum of the negative X_m exponents in W to the left of W_i. The resulting word $W_0'W_1' \dots W_l'$ between X_m^{pos} and X_m^{neg} is called the *core* of W and denoted $\textsc{Core}(W)$.

The construction of the core of a word, is at the heart of the process that produces the Thompson normal form.

Remark 10.8 (Producing the Thompson normal form)**.** Let W be a reduced word and let $W' = X_m^{pos}\textsc{Core}(W)X_m^{neg}$ be the word representing the same element of $\mathbf{F}$ produced by the process described above. If the first letter of W' is X_m, the last letter is X_m^{-1} and $\min(\textsc{Core}(W)) > m+1$ then we can cancel an X_m and an X_m^{-1} and downshift $\textsc{Core}(W)$ to produced an equivalent word whose core has a smaller minimal subscript. We can repeat this process until the extra condition required by the normal form is satisfied with respect to the subscript m. At this stage we repeat this entire process on the new core, the down-shifted $\textsc{Core}(W)$. After a finite number of iterations, the end result is an equivalent word in Thompson normal form.

From the description of the rewriting process, the following proposition should be obvious.

Proposition 10.9 (Increasing subscripts)**.** *If W is word with $\min(W) = n$ and a non-trivial Thompson normal form W', then $\min(W')$ is least n. In particular, when $\min(W) > m$, the words W and X_m^e, e nonzero, represent distinct elements of $\mathbf{F}$.*

Example 10.10. Consider the following word:

$$W = (X_2^{-3}X_5^2)X_0^4(X_1^5X_3^{-2})X_0^{-1}(X_1^7)X_0^2(X_3X_4)$$

It has $\min(W) = 0$, $pos = 6$, $neg = -1$. Pulling the syllables with minimal subscripts to the front and back produces the equivalent word:

$$W' = X_0^6(X_8^{-3}X_{11}^2)(X_3^5X_5^{-2})(X_4^7)(X_4X_5)X_0^{-1}$$

with $\textsc{Core}(W) = X_8^{-3}X_{11}^2X_3^5X_5^{-2}X_4^8X_5$. Note that we needed to combine two syllables in order for the core to be in syllable normal form. The process of reducing this to Thompson normal form would further

cancel an initial X_0 with a terminal X_0^{-1} and down shift the core because $\min(\textsc{Core}(W)) = 3 > 1+1$. The new word is:

$$W' = X_0^5(X_7^{-3}X_{10}^2X_2^5X_4^{-2}X_3^8X_4)$$

and the new core is $X_7^{-3}X_{10}^2X_2^5X_4^{-2}X_3^8X_4$.

The key technical result we need is the following.

Lemma 10.11. *The W is an odd word then* $\textsc{Core}(W)$ *is an odd word with strictly fewer syllables.*

Proof. Since the shift of an odd word remains odd, we only need to show that for each i the last subscript of W'_{i-1} and first subscript of W'_i have opposite parity. First note that in the original word W, the last subscript of W_{i-1} and the first subscript of W_i have the same parity since they are separated by the syllable $X_m^{e_i}$. The key observation is that the subscripts in the words on either side of $X_m^{e_i}$ are shifted upward by the same amount except that one of the two is shifted an additional $|e_i|$ times, ensuring that the newly adjacent subscripts have opposite parity. Which side receives the additional shifting is, of course, determined by the sign of e_i. $\square$

It is an interesting question for which classes of sets A and E, (E, A)-words are always non-trivial. We do not study this question; we just need the following proposition.

Theorem 10.12 (Odd words in $\mathbf{F}$)**.** *Every odd word represents a non-trivial element of* $\mathbf{F}$. *As a consequence, so long as E and A are selected from $\mathbb{Z}_{odd}$, every (E, A)-word represents a non-trivial element of* $\mathbf{F}$.

Proof. Let $W = X_{n_1}^{e_1}X_{n_2}^{e_2}\dots X_{n_k}^{e_k}$ be an odd word. The proof is by induction on k. If $k = 1$ then $W = X_n^e$ is in Thompson normal form and non-trivial by Theorem 10.4. So assume this result is true for all odd words with fewer than k syllables. As described in Definition 10.7, W is equivalent in $\mathbf{F}$ to the word $X_m^{pos}\textsc{Core}(W)X_m^{neg}$ where $m = \min(W)$. If this word were trivial then $\textsc{Core}(W)$ and $X_m^{-(pos+neg)}$ would represent the same element of $\mathbf{F}$. We split this into two cases. When $pos + neg$ is nonzero, we note that $\min(\textsc{Core}(W)) > m$ by construction and thus these words represent different elements of $\mathbf{F}$ by Proposition 10.9. On the other hand, when $pos + neg$ is zero, the question becomes whether $\textsc{Core}(W)$ itself can be trivial in $\mathbf{F}$. This too is impossible by Lemma 10.11 and the induction hypothesis. Therefore W represents a non-trivial element of $\mathbf{F}$. $\square$

Remark 10.13 (Not injective)**.** It is important to realize that just because there are no odd words in the kernel does not mean that the homomorphism $\textsc{Free}(\mathbb{X}) \to \mathbf{F}$ is injective on the elements represented by odd words. In fact, it is easy to find pairs of odd words, such as $X_1^{-1}X_2X_1$ and X_3, that represent the same element in $\mathbf{F}$. Part of the explanation here is that even though W_1 and W_2 odd implies that W_1^{-1} is odd, the word $W_1^{-1}W_2$ need not be. The injectivity fails even for the homomorphism $\textsc{Free}(\mathbb{A}) \to \mathbf{F}$.

Now we are close to claim that $\mathbf{F}$ satisfies the conditions of Theorem 2.4, but for that, first, we need to introduce the height function

Definition 10.14. Let $w = X_{n_0}^{e_0}X_{n_1}^{e_1}\dots X_{n_k}^{e_k}X_{m_l}^{-f_l}\dots X_{m_0}^{-f_0}$ be a Thompson normal form of $W \in \mathbf{F}$. Then $h(W) = (e_0+\dots+e_k)+(f_0+\dots+f_l)$

Proposition 10.15. *With $\eta = X_0, \xi = X_1$, the group $\mathbf{F}$ satisfies property (D).*

Proof. It is clear that the function $h : \mathbf{F} \to \mathbb{N} \cup \{0\}$ is subadditive. Let $g \in \mathbf{F}, W = X_{n_0}^{e_0}X_{n_1}^{e_1}\dots X_{n_k}^{e_k}X_{m_l}^{-f_l}\dots X_{m_0}^{-f_0}$ be the Thompson normal form of g. For any $\epsilon, \delta \in \mathbb{Z}$, let $W(\epsilon, \delta) = W\eta^{\delta}\xi^{\epsilon}$.

If $h(g\eta^{\delta}\xi^{-p}) > h(g)+100$ for all $\delta \in (-100, 100)$ then we have nothing to prove. So let δ_0 be the biggest number in the interval $(-100, 100)$ such that $h(W(-p, \delta_0) \le h(g) + 100(\dagger)$.

We will consider several cases:

Case 1: $m_0 > 0$.

Then necessarily $\delta_0 < 0, n_0 = 0$, moreover, the inequlaity $(\dagger)$ can hold only in the following cases

a) when $W(-p, \delta_0) = X_{n_0}^{e_0}X_{n_1}^{e_1}\dots X_{n_k}^{e_k}X_{n_k}^{-p}X_{m_l}^{-f_l}\dots X_{m_0}^{-f_0}X_0^{\delta_0}(\dagger\dagger)$ and $n_k > m_l$.

Then for any $-100 \le \delta < \delta_0$ by we obtain that

$$W(-p, \delta) = X_{n_0}^{e_0}X_{n_1}^{e_1}\dots X_{n_k}^{e_k}X_{n}^{-p}X_{m_l}^{-f_l}\dots X_{m_0}^{-f_0}X_0^{\delta}$$

where $n > n_k$ thus we get $h(W(-p, \delta)) \ge h(W) - 100 + p > h(W) + 100$ hence part (i) of condition (D) is verified.

For part (ii), notice that (for same $\delta \in [-100, \delta_0)$)

$$W(p, \delta) = X_{n_0}^{e_0}X_{n_1}^{e_1}\dots X_{n_k}^{e_k}X_{n}^{p}X_{m_l}^{-f_l}\dots X_{m_0}^{-f_0}X_0^{\delta_{n+1}}$$

therefore we have $h(W(p, \delta)) \ge h(W) - 100 + p > h(W) + 100$.

Finally, notice that under the assumption (††),

$$W(p,\delta_0) = X_{n_0}^{e_0} X_{n_1}^{e_1} \dots X_{n_k}^{e_k} X_{n_k}^{p} X_{m_l}^{-f_l} \dots X_{m_0}^{-f_0} X_0^{\delta_0}$$

hence $h(W(p,\delta_0)) \geq h(W) - 100 + p > h(W) + 100$.

b) when

$$W(-p,\delta_0) = X_{n_0}^{e_0} \dots X_{n_i}^{e_i} \dots X_{n_k}^{e_k} X_{m_l}^{-f_l} \dots X_{m_j}^{-f_j} X_n^{-p} X_{m_{j-1}}^{-f_{j-1}} \dots X_{m_0}^{-f_0} X_0^{\delta_{n+1}}$$

and $n_i = n, i < k$:

Then since $-p < -400, -100 < \delta_0 < 100$, the inequality $h(W(-p,\delta_0)) \leq h(W)$ yields that $min\{n_{i+1}, m_j\} > n + 200$. But then, as in part a), by letting $\delta \in [-100, \delta_0)$, we obtain that $h(W(-p,\delta)) > h(W) + 100$. Also, as in Case a), we obtain $h(W(p,\delta)) > h(W) + 100$ for all $\delta \in (-100, \delta_0]$.

Case 2: $m_0 = 0$. Then, we observe that if $-m_0 + \delta_0 > 0$, then $h(W(-p,\delta_0) > h(W) + 300$. But if $-m_0 + \delta_0 \leq 0$ then this case is no different from the previous case.

Case 3: $W = X_{n_0}^{e_0} X_{n_1}^{e_1} \dots X_{n_k}^{e_k}$, *i.e. the negative part of the Thompson normal form is absent.*

For the inequality $h(g\eta^{\delta}\xi^{-p}) > h(g) + 100$, this case is not different from the previous cases, and for the inequality $h(g\eta^{\delta}\xi^{p}) > h(g) + 100$ we observe that it holds for all $\delta \in [-100, 100]$. □

Theorem 10.16. **F** *satisfies all conditions of Theorem 2.4. therefore it is non-amenable.*

Proof. It is wellknown that $\mathbf{F}/[\mathbf{F},\mathbf{F}]$ is isomorphic to $\mathbb{Z}^2$. (See [CFP]). We choose $\eta = X_0, \xi = X_1$. Then condition (C) follows from Theorem 10.12. and condition (D) follows from Proposition 10.15.

11. Unbalanced Words.

In this section we will introduce a class of semi-odd words to emphasize some difficulties in applying Theorem 2.2. directly to **F**. This class of words is a generalization of the type of words we considered in Proposition 10.15.

It is a major problem to find *rich* and *structured* class of reduced words in the alphabet $\{X_0, X_0^{-1}, X_1, X_1^{-1}\}$ which may lie arbitrarily deep in the derived series of Free($\mathbb{A}$) but do not vanish in **F**. We do not make an attempt here to define the terms *rich* and *structured* since, formally, we do not need them. Of course, examples of such

words are not difficult to construct, if that is the purpose; i.e. given any natural number d, it is easy (for example, using the description of $\mathbf{F}$ as a group of piece-wise linear maps) to construct a non-trivial word in $\mathbf{F}$ which lies in the d-th derived subgroup of $\textsc{Free}(\mathbb{A})$. For our purpose, words constructed this way are not rich enough. On the other hand, since the commutator subgroup $[\mathbf{F},\mathbf{F}]$ is simple, for any element in $z \in [\mathbf{F},\mathbf{F}]$ and for any natural number d one can represent z as a reduced word in the alphabet $\{X_0, X_0^{-1}, X_1, X_1^{-1}\}$ which lies in the d-th derived subgroup of $\textsc{Free}(\mathbb{A})$. However, these representations are not structured enough to help us in proving non-amenability.

Definition 11.1 (Semi-(E, A)-words, semi-odd words)**.** Let $W = X_{n_1}^{e_1} X_{n_2}^{e_2} \dots X_{n_k}^{e_k}$ be a word over $\mathbb{X}$ in syllable normal form and let A and E be subsets of $\mathbb{Z}^*$. We call W a semi-(E, A)-word if the exponents lie in E and the adjacent subscript differences lie in A with at most one exception (i.e. either one exponent not in E or one adjacent subscript difference not in A but not both). If $E = A = \mathbb{Z}_{\text{odd}}$, W is called a *semi-odd word.*

Example 11.2. Given any non-trivial reduced word W over $\mathbb{X}$ in syllable normal form, we define a finite sequence of words $W_0, W_1, W_2, \dots W_k$ starting with $W_0 = W$, defining $W_{i+1} = \textsc{Core}(W_i)$, and stopping when W_k is a single syllable $X_n^e, e \in \mathbb{Z}$. The sequence $(W_n)_{n\geq 0}$ will be called *the core sequence of* W. Here are two examples. If U is the word U_0 listed below, then we get the following sequence:

$$\begin{array}{rcl}
U_0 &=& X_0^{-5}X_1^{-5}X_0^{-7}X_1^{3}X_0X_1^{-5}X_0^{2}X_1^{-5}X_0^{3}X_1X_0^{-7}X_1^{-5}X_0X_1^{3},\\
U_1 &=& \textsc{Core}(U_0) = X_6^{-5}X_{13}^{3}X_{12}^{-5}X_{10}^{-5}X_7X_2^{-5}X_1^{3},\\
U_2 &=& \textsc{Core}(U_1) = X_9^{-5}X_{16}^{3}X_{15}^{-5}X_{13}^{-5}X_{10}X_5^{-5},\\
U_3 &=& \textsc{Core}(U_2) = X_9^{-5}X_{16}^{3}X_{15}^{-5}X_{13}^{-5}X_{10},\\
U_4 &=& \textsc{Core}(U_3) = X_{21}^{3}X_{20}^{-5}X_{18}^{-5}X_{15},\\
U_5 &=& \textsc{Core}(U_4) = X_{22}^{3}X_{21}^{-5}X_{19}^{-5},\\
U_6 &=& \textsc{Core}(U_5) = X_{22}^{3}X_{21}^{-5}, \text{ and}\\
U_7 &=& \textsc{Core}(U_6) = X_{22}^{3}
\end{array}$$

And if V is the word V_0 listed below, then we have:

$$\begin{array}{rcl}
V_0 &=& X_2X_1X_2^{-5}X_1X_3X_4^{-5}X_1^{3},\\
V_1 &=& \textsc{Core}(V_0) = X_7X_6^{-5}X_6X_7^{-5} = X_7X_6^{-4}X_7^{-5},\\
V_2 &=& \textsc{Core}(V_1) = X_7X_{11}^{-5}, \text{ and}\\
V_3 &=& \textsc{Core}(V_2) = X_{11}^{-5}
\end{array}$$

Notice that both of the words U and V are semi-odd-words, and all the terms in their sequences, i.e. the words $U_1, \dots, U_7, V_1, \dots, V_3$

remain so. Both of the sequences turn from semi-odd into odd at some point and remain odd from that moment further; the word U_6 (and U_7) is an unbalanced odd word, so is the word V_7.

Now we will introduce special classes of semi-odd words which are never trivial in $\mathbf{F}$.

Definition 11.3 (Unbalanced words)**.** A set $E \subset \mathbb{Z}_{\text{odd}}$ is called unbalanced if there are no $x, y \in E$ such that $x = -y$. Let $W = X_{n_1}^{e_1} X_{n_2}^{e_2} \dots X_{n_k}^{e_k}$ be a word over $\mathbb{X}$ in syllable normal form, and $e_i \in E, 1 \leq i \leq k$. If E is unbalanced then W is called unbalanced.

In general, it is an interesting question for which $E, A \subseteq \mathbb{Z}^*$ the semi-(E, A)-words are always non-trivial in F. We do not explore this question here but restrict ourselves to the following proposition which demonstrates a very interesting phenomenon occurring in $\mathbf{F}$.

Proposition 11.4. *If W is an unbalanced semi-odd word then $W \neq 1 \in \mathbf{F}$.*

Proof. Let us assume the opposite. Let $W_0, W_1, \dots, W_s$ be the core sequence of W where $W_0 \equiv W, W_s \equiv 1$, and W_{s-1} is not a trivial word. Then $W_{s-1} \equiv W' X_p^{e'} X_m^{e} X_q^{e''} W''$ where $m = min(W_{s-1})$ and precisely one of the numbers $e, p - m, q - m$ is a non-zero even number and the other two are odd numbers. In either case we obtain that $e' = -e''$ which contradicts the assumption that W is unbalanced. $\square$

12. What goes wrong in solvable groups?

In this section we will discuss the comparison between $\mathbf{F}$ and free solvable groups on $n \geq 2$ generators of derived length $d \geq 2$.

Let $G(n, d)$ be a free solvable group on n generators of derived length d. The case $n = 2$ is already very interesting to us.

We would like to remark that from geometrical and combinatorial point of view, the group $G(2, d)$, for $d \geq 2$ is not any easier than $\mathbf{F}$; and it is not a less interesting example for the theory of amenable groups. In fact, $\mathbf{F}$ is often associated with the wreath product $\mathbb{Z} \wr \mathbb{Z}$ so it can be placed in between $\mathbb{Z} \wr \mathbb{Z}$ and the free metaabelian group $G(2, 2)$ despite not being solvable (more close to $\mathbb{Z} \wr \mathbb{Z}$, since $\mathbf{F}$ has many subgroups isomorphic to $\mathbb{Z} \wr \mathbb{Z}$, and although these subgroups are of infinite index,

they are very *close to the surface* of **F**). The closeness of **F** to $\mathbb{Z} \wr \mathbb{Z}$ and/or to $G(2,2)$ is a much more series issue than it may sound. **F** is full of commuting elements, and especially, from the point of view of our approach, any reasonable class of words potentially applicable to **F** contains commutators of higher and higher depth and may very quickly vanish in **F**.

However, notice that condition (C) totally fails in $\mathbb{Z} \wr \mathbb{Z}$! Thus in our struggle of trying to make some way for **F** out of the minefield of commuting elements, we have already left $\mathbb{Z} \wr \mathbb{Z}$, close friend of **F** , far behind us!

It is not difficult to notice, however, that the group $G(2,2)$ is still with us; condition (C) still holds in $G(2,2)$! It is indeed very difficult to come up with a class of words potentially applicable to **F** but not to $G(2,2)$ (so one often plays on a very thin line of trying to prove non-amenability of **F** but at the same time being extremely careful of not proving a false claim that "free solvable groups are non-amenable")

To draw more parallels, we introduce the notions of odd and semi-odd words for $G(2,d)$:

Definition 12.1. Let $G(2,d)$ be generated by x and y. A reduced word $W(x,y)$ is called odd iff all exponents of x and y are odd. $W(x,y)$ is called semi-odd if all but maybe one of the exponents is odd. $W(x,y)$ is called unbalanced in x (in y) if no two exponents of x (of y) in W add up to zero. Finally, W is called unbalanced if it is unbalanced in at least one of the letters x and y.

We would like to conclude with the following remarks

Remark 12.2 (unbalanced words in free solvable groups)**.** Like in **F**, odd words in $G(2,d), d \geq 2$ are never trivial, and semi-odd words could be trivial. The unbalanced type condition (in more general sense) implicitly plays an important role in the proof of Proposition 10.15, which proves condition (D) for **F** (we always avoid one "bad" value of the exponent δ_{n+1}). However, this condition turns out to be uneffective in free solvable groups. Indeed, let

$$w_0 = ab^{-1}a^{-1}b^4ab^{-1}a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}ab^4a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}$$

where $\{a,b\}$ is the standard generating set of $G(2,d)$.
One can see that w_0 can be written as

$$w_0 = [a,b^{-1}][b^3,a][a,b^2][b,a][b^{-1},a][a,b^3][b^2,a][a,b]$$

and notice that the first commutator is the inverse of the 5th, the 2nd is the inverse of the 6th, the 3rd is the inverse of the 7th, and finally, the 4th is the inverse of the 8th. So w_0 lies in the second commutator subgroup thus is equal to the identity element of $G(2,2)$. On the other hand, w_0 is unbalanced in b. There are examples of unbalanced semi-odd words as well which become trivial in $G(2,d)$.

It turns out even some modification of unbalanced condition still fails in $G(2,d), d \geq 2$, and condition (D) is a condition of this type.

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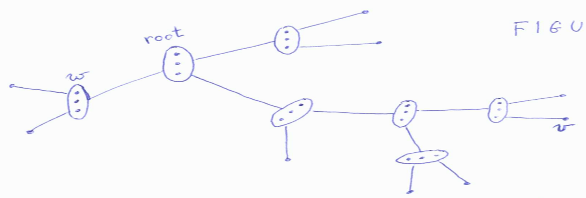

FIGURE 1.

This Generalized Binary Tree has 7 internal vertices and 9 end vertices. v is an endvertex of level 4. w is an internal vertex of level 1.

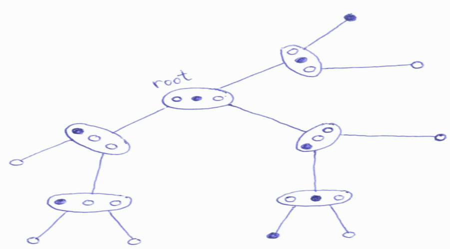

FIGURE 2.

This GBT is black; all internal vertices are black. The end vertices may be of either color.

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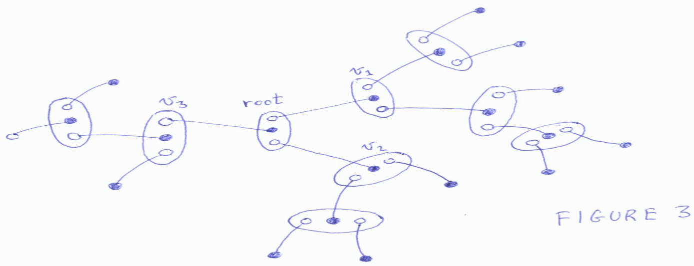


FIGURE 3

Labeled Black GBT.
The end vertices beyond v_1 and v_2 are black; the end vertices beyond v_3 may still have a white color.

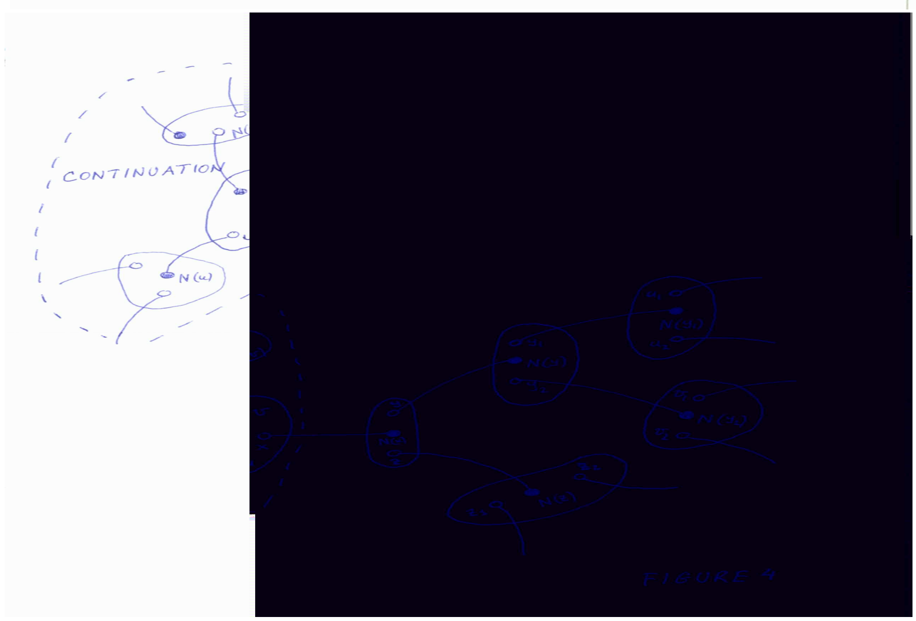


FIGURE 4

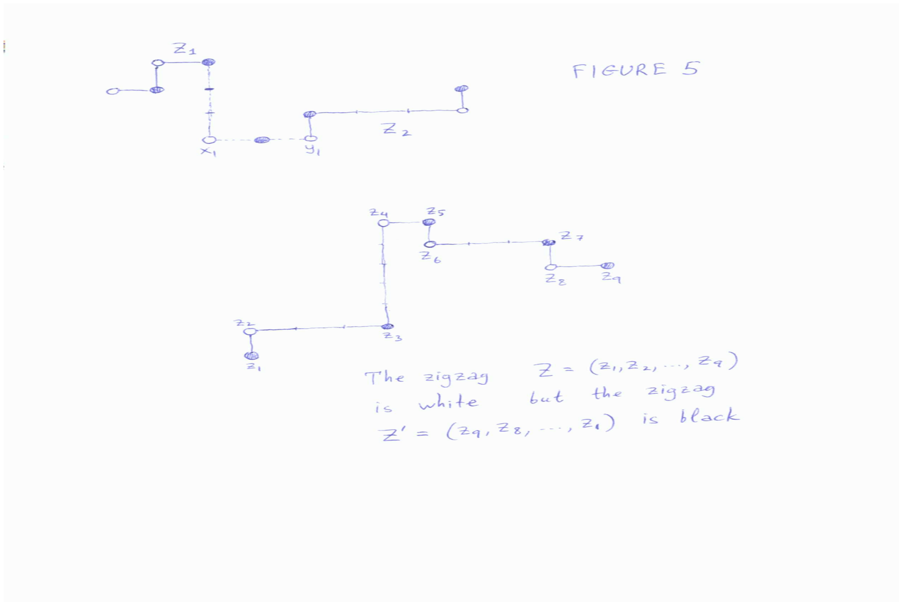


FIGURE 5
Z_1
Z_2
x_1
y_1
z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 z_9
The zigzag $Z = (z_1, z_2, \dots, z_9)$
is white but the zigzag
$Z' = (z_9, z_8, \dots, z_1)$ is black

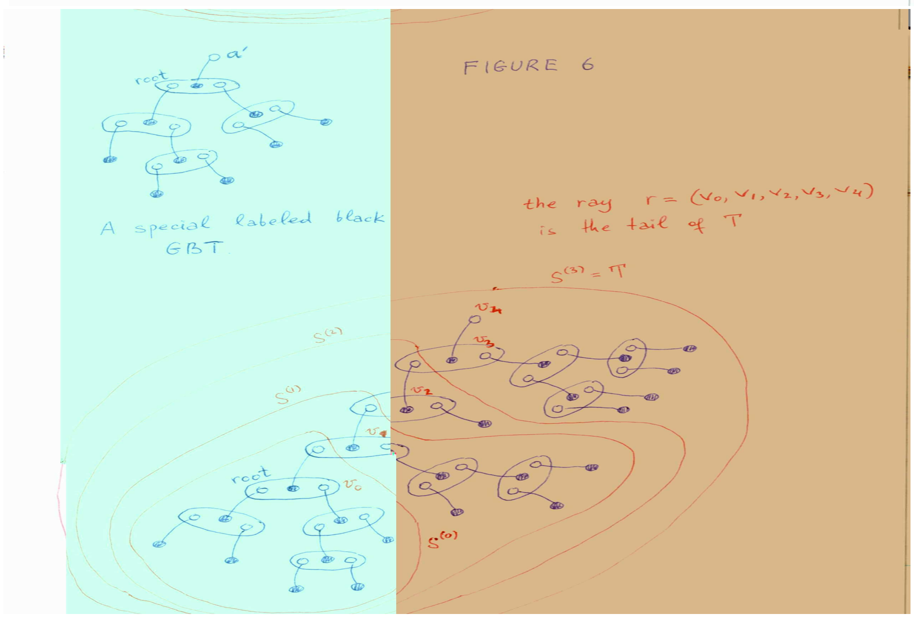
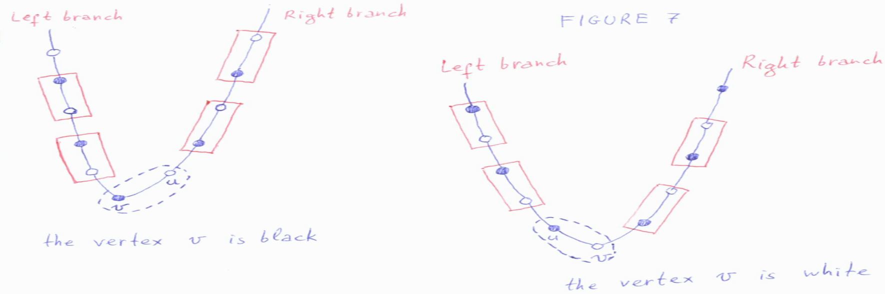


FIGURE 6
a'
root
A special labeled black GBT.
the ray $r = (v_0, v_1, v_2, v_3, v_4)$
is the tail of T
$S^{(3)} = T$
$S^{(2)}$
$S^{(1)}$
$S^{(0)}$
v_0 v_1 v_2 v_3 v_4
root

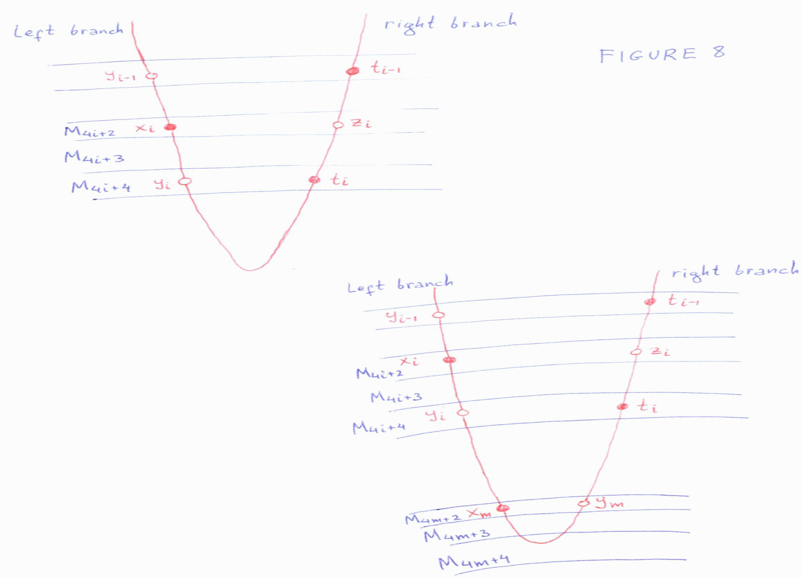
FIGURE 7



1) white and black elements inside rectangular boxes are ξ-partners

2) the ξ-partners of v and u lie in $\partial_K F$.

FIGURE 8



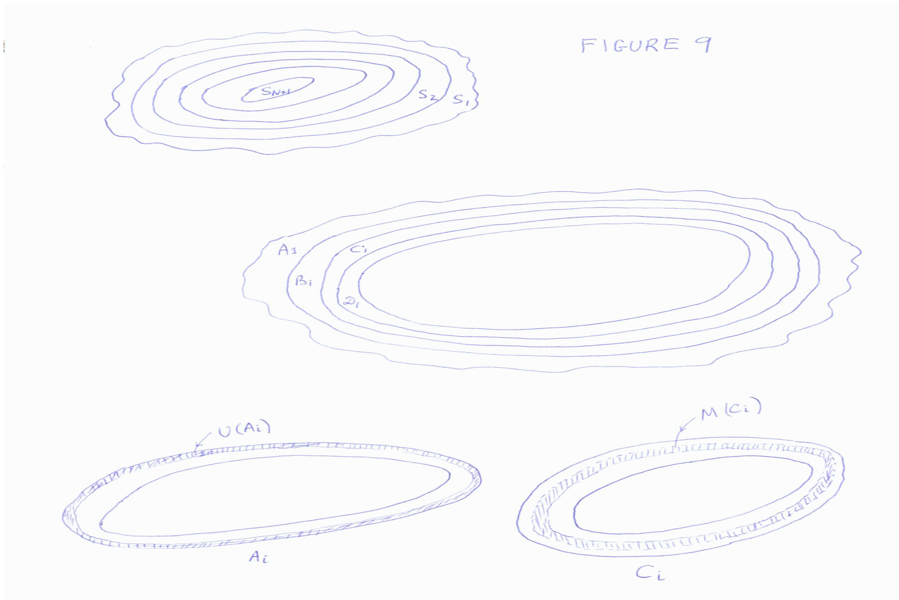


FIGURE 9
S N+1
S2
S1
A1
C1
B1
D1
U(Ai)
Ai
M(Ci)
Ci